

Improved Spanning Ratio for Low Degree Plane Spanners^{*}

Prosenjit Bose, Darryl Hill, and Michiel Smid

School of Computer Science, Carleton University, Ottawa, Canada

Abstract. We describe an algorithm that builds a plane spanner with a maximum degree of 8 and a spanning ratio of ≈ 4.414 with respect to the complete graph. This is the best currently known spanning ratio for a plane spanner with a maximum degree of less than 14.

1 Introduction

Let P be a set of n points in the plane. Let G be a weighted geometric graph on vertex set P , where edges are straight line segments and are weighted according to the Euclidean distance between their endpoints. Let $\delta_G(p, q)$ be the sum of the weights of the edges on the shortest path from p to q in G . If, for graphs G and H on the point set P , where G is a subgraph of H , for every pair of points p and q in P , $\delta_G(p, q) \leq t \cdot \delta_H(p, q)$ for some real number $t > 1$, then G is a t -spanner of H , and t is called the *spanning ratio*. H is called the *underlying graph* of G . In this paper the underlying graph is the Delaunay triangulation or the complete graph.

The L_1 -Delaunay triangulation was first proven to be a $\sqrt{10}$ -spanner by Chew [1]. Dobkin *et al.* [2] proved that the L_2 -Delaunay triangulation is a $\frac{1+\sqrt{5}}{2}\pi$ -spanner. This was improved by Keil and Gutwin [3] to $\frac{2\pi}{3\cos(\frac{\pi}{6})}$, and finally taken to its currently best known spanning ratio of 1.998 by Xia [4].

The Delaunay triangulation may have an unbounded degree. High degree nodes can be detrimental to real world applications of graphs. Thus there has been research into bounded degree plane spanners. We present a brief overview of some of the results in Table 1.

Bounded degree plane spanners are often obtained by taking a subset of edges of an existing plane spanner and ensuring that it has bounded degree, while maintaining spanning properties. We note how in Table 1 that all of the results are subgraphs of some variant of the Delaunay triangulation.

Our contribution is an algorithm to construct a plane spanner of maximum degree 8 with a spanning ratio of ≈ 4.41 . This is the lowest spanning ratio of any graph of degree less than 14.

The rest of the paper is organized as follows. In Section 2 we describe how to select a subset of the edges of the Delaunay triangulation $DT(P)$ to form

^{*} This work was partially supported by the Natural Sciences and Engineering Research Council of Canada (NSERC) and by the Ontario Graduate Scholarship (OGS).

Paper	Degree	Stretch Factor
Bose <i>et al.</i> [5]	27	$(\pi + 1)C_{DT} \approx 8.27$
Li & Wang [6]	23	$(1 + \pi \sin(\frac{\pi}{4}))C_{DT} \approx 6.44$
Bose <i>et al.</i> [7]	17	$(\frac{1+\sqrt{3}+3\pi}{2} + 2\pi \sin(\pi/12))C_{DT} \approx 23.58$
Kanj <i>et al.</i> [8]	14	$(1 + \frac{2\pi}{14 \cos(\pi/14)})C_{DT} \approx 2.92$
Bose <i>et al.</i> [9]	7	$(\frac{1}{1-2 \tan(\pi/8)})C_{DT} \approx 11.65$
Bose <i>et al.</i> [9]	6	$(\frac{1}{(1-\tan(\pi/7)(1+1/\cos(\pi/14)))})C_{DT} \approx 81.66$
Bonichon <i>et al.</i> [10]	6	6
Bonichon <i>et al.</i> [11]	4	$\sqrt{4 + 2\sqrt{2}}(19 + 29\sqrt{2}) \approx 156.82$
This paper	8	$(1 + \frac{2\pi}{6 \cos(\pi/6)})C_{DT} \approx 4.41$

C_{DT} is the spanning ratio of the Delaunay triangulation, currently < 1.998 [4]

Table 1: Known results for bounded degree plane spanners.

the graph $D8(P)$. In Section 3 we prove that $D8(P)$ has a maximum degree of 8. In Section 4 we bound the spanning ratio of $D8(P)$ with respect to $DT(P)$. Since $DT(P)$ is a spanner of the complete Euclidean graph, this makes $D8(P)$ a spanner of the complete Euclidean graph as well.

2 Building $D8(P)$

Given as input a set P of n points in the plane, we present an algorithm for building a bounded degree plane graph with maximum degree 8 and spanning ratio bounded by a constant, which we denote as $D8(P)$. The graph denoted $D8(P)$ is constructed by taking a subset of the edges of the Delaunay triangulation of P , denoted $DT(P)$.

We assume general position of P ; i.e., no three points are on a line, no four points are on a circle, and no two points form a line with slope 0, $\sqrt{3}$ or $-\sqrt{3}$.

The space around each vertex p is partitioned by *cones* consisting of 6 equally spaced rays from p . Thus each cone has an angle of $\pi/3$. See Figure 1a. We number the cones starting with the topmost cone as C_0 , then number in the clockwise direction. Cone arithmetic is modulo 6. By our general position assumption we note that no point of P lies on the boundary of a cone.

We introduce a distance function known as the *bisector distance*, which is the distance from p to the orthogonal projection of q onto the bisector of C_i^p , where $q \in C_i^p$. We denote this length $[pq]$. Any reference made to distance is to the bisector distance, unless otherwise stated.

Definition 1. Let $\{q_0, q_1, \dots, q_{d-1}\}$ be the sequence of all neighbours of p in $DT(P)$ in consecutive clockwise order. The neighbourhood N_p , with apex p , is the graph with the vertex set $\{p, q_0, q_1, \dots, q_{d-1}\}$ and the edge set $\{(p, q_j)\} \cup \{(q_j, q_{j+1})\}, 0 \leq j \leq d-1$, with all values mod d . The edges $\{(q_j, q_{j+1})\}$ are called canonical edges. N_i^p is the subgraph of N_p induced by all the vertices of N_p in C_i^p , including p . This is called the **cone neighbourhood** of p . See Figure 1b.

The algorithm *ConstructD8(P)* takes as input a point set P and returns the bounded degree graph $D8(P)$, with vertex set P and edge set E . The algorithm

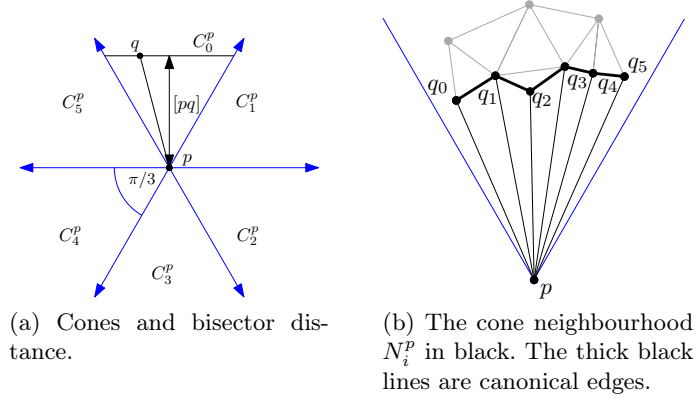


Fig. 1: Preliminaries.

calls two subroutines. *AddIncident()* selects a set of edges E_A . For each edge (p, r) of E_A , we call *AddCanonical* (p, r) and *AddCanonical* (r, p) which add edges to the set E_{CAN} . Both E_A and E_{CAN} are a subset of the edges in $DT(P)$. The final graph $D8(P)$ consists of the vertex set P and the union of edge sets E_A and E_{CAN} .

We present the algorithm here:

Algorithm: ConstructD8(P)

INPUT: Set P of n points in the plane.

OUTPUT: $D8(P)$: spanning subgraph of $DT(P)$.

- Step 1: Compute the Delaunay triangulation $DT(P)$ of the point set P .
- Step 2: Sort all the edges of $DT(P)$ by their bisector length, into a set L , in non-decreasing order.
- Step 3: Call the function *AddIncident* (L) with L as the argument. *AddIncident()* selects and returns the subset E_A of the edges of L .
- Step 4: For each edge (p, r) in E_A in sorted order call *AddCanonical* (p, r) and *AddCanonical* (r, p) , which add edges to the set E_{CAN} .
- Step 5: Return $D8(P) = (P, E_A \cup E_{CAN})$.

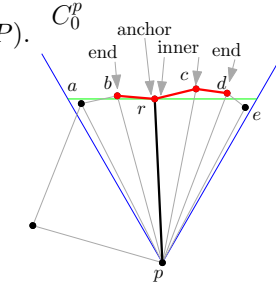


Fig. 2: The graph Can_0^p , based on $(p, r) \in E_A$, in red. Vertex r is the anchor, d and b are end vertices, and c and a are inner vertices.

Algorithm: AddIncident(L)

INPUT: L : set of edges of $DT(P)$ sorted by bisector distance.

OUTPUT: E_A : a subset of edges of $DT(P)$.

- Step 1: Initialize the set $E_A = \emptyset$.
- Step 2: For each $(p, q) \in L$, in non-decreasing order, do:

- (a) Let i be the cone of p containing q . If E_A has no edges with endpoint p in N_i^p , and if E_A has no edges with endpoint q in N_{i+3}^q , then we add (p, q) to E_A .

Step 3: return E_A .

The next algorithm requires the following definition:

Definition 2. Let $Can_i^{(p,r)}$ be the subgraph of $DT(P)$ consisting of the ordered subsequence of canonical edges (s, t) of N_i^p in clockwise order around apex p such that $[ps] \geq [pr]$ and $[pt] \geq [pr]$. We call $Can_i^{(p,r)}$ a canonical subgraph. A vertex that is the first or last vertex of $Can_i^{(p,r)}$ is called an end vertex of $Can_i^{(p,r)}$. A vertex that is not the first or last vertex in $Can_i^{(p,r)}$ is called an inner vertex of $Can_i^{(p,r)}$. Vertex r is called the anchor of $Can_i^{(p,r)}$. See Fig. 2.

Algorithm: AddCanonical(p,r)

INPUT: (p, r) , an edge of E_A .

OUTPUT: A set of edges that are a subset of the edges of $DT(P)$. All edges generated by calls to *AddCanonical()* form the set E_{CAN} .

Step 1: Without loss of generality, let $r \in C_0^p$.

Step 2: If there are at least three edges in $Can_0^{(p,r)}$, then for every canonical edge (s, t) in $Can_0^{(p,r)}$ that is not the first or last edge in the ordered subsequence of canonical edges $Can_0^{(p,r)}$, we add (s, t) to E_{CAN} .

Step 3: If the anchor r is the first or last vertex in $Can_0^{(p,r)}$, and there is more than one edge in $Can_0^{(p,r)}$, then add the edge of $Can_0^{(p,r)}$ with endpoint r to E_{CAN} . See Fig. 4b.

Step 4: Consider the first and last canonical edge in $Can_0^{(p,r)}$. Since the conditions for the first and last canonical edge are symmetric, we only describe how to process the last canonical edge (y, z) . There are three possibilities.

- (a) If $(y, z) \in N_5^z$ we add (y, z) to E_{CAN} . See Fig. 4c.
- (b) If $(y, z) \in N_4^z$ and N_4^z does not have an edge with endpoint z in E_A , then we add (y, z) to E_{CAN} . See Fig. 4d
- (c) If $(y, z) \in N_4^z$ and there is an edge with endpoint z in $E_A \cap N_4^z \setminus (y, z)$, then there is exactly one canonical edge of z with endpoint y in N_4^z . We label this edge (w, y) and add it to E_{CAN} . See Fig. 4e.

3 D8(P) has Maximum Degree 8

To prove $D8(P)$ has a maximum degree of 8 we use a simple charging scheme. We charge each edge (p, q) of $D8(P)$ once to p and once to q . Thus the total charge on a vertex is equal to the degree of that vertex. To help track the number of charges on a vertex, each charge is associated with a specific cone, which may

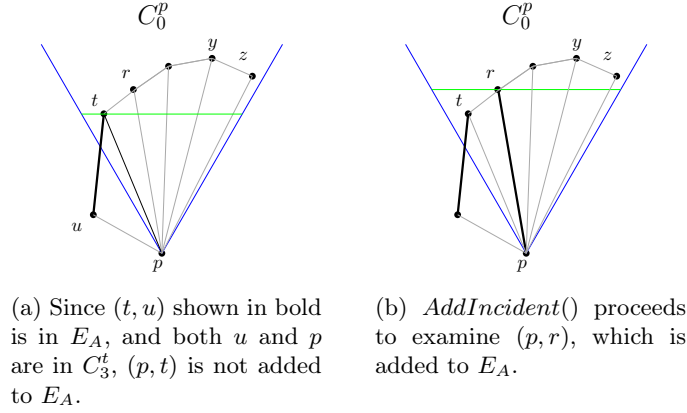


Fig. 3: $AddIncident(L)$ selects edge (p, r) for E_A .

not be the cone containing the edge. We show that a cone can be charged at most twice, and that for any vertex p of P , at most two cones of p can be charged twice, while the remaining cones are charged at most once, which yields our maximum degree of 8.

Sections 3.1, 3.2, 3.3, and 3.4 identify the different types of cones and their properties. Sections 3.5 and 3.6 detail the charging scheme. Section 3.7 proves the maximum degree of $D8(P)$.

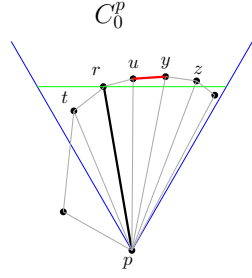
3.1 Cone Types

Definition 3. For an arbitrary cone neighbourhood N_i^P we define the region of N_i^P as the polygonal region bounded by the canonical edges of N_i^P and the first and last edge of N_i^P with endpoint p . See Figure 5a.

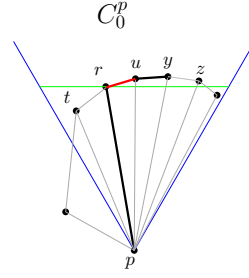
Definition 4. Let (a, b) be the first edge and let (y, z) be the last edge in a canonical subgraph $Can_i^{(p, r)}$ (Definition 2). We define the region of $Can_i^{(p, r)}$ as the polygonal region bounded by the canonical edges of N_i^P between (a, b) and (y, z) inclusive, and the edges (p, a) and (p, z) . See Figure 5b.

We provide the following definitions regarding the placement of cones in regions. Both of the following definitions also extend to regions of a cone neighbourhood.

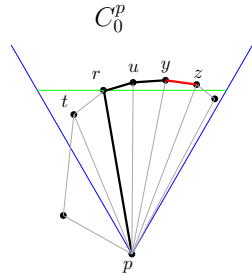
Definition 5. Let $B(s, \epsilon)$ be a ball with center s and radius $\epsilon > 0$. Consider a cone C_j^s of a point s in $Can_i^{(p, r)}$. If there exists an $\epsilon > 0$ such that $B(s, \epsilon) \cap C_j^s$ is inside the region of $Can_i^{(p, r)}$, then we call this an internal cone of $Can_i^{(p, r)}$. Alternatively we say C_j^s is in $Can_i^{(p, r)}$.



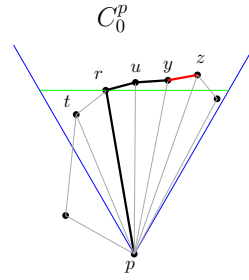
(a) Edges added to E_{CAN} in Step 2.



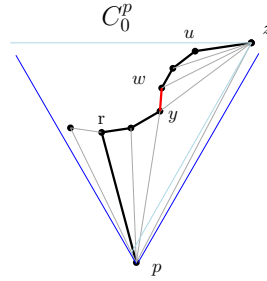
(b) Edge added to E_{CAN} in Step 3.



(c) Edge added to E_{CAN} in Step 4a.



(d) Edge added to E_{CAN} in Step 4b.



(e) Edge added to E_{CAN} in Step 4c.

Fig. 4: AddCanonical(p, r)

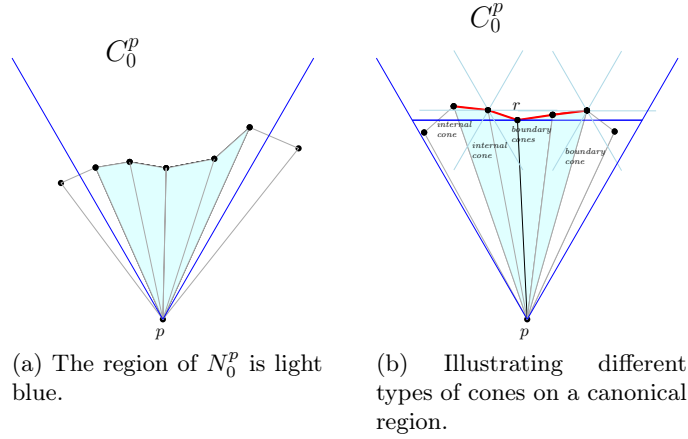


Fig. 5: Regions and cones.

Definition 6. Consider a cone C_j^s of a point s in $\text{Can}_i^{(p,r)}$. If for all $\epsilon > 0$, $B(s, \epsilon) \cap C_j^s$ is partially but not entirely in the region of $\text{Can}_i^{(p,r)}$, then we call this a boundary cone of $\text{Can}_i^{(p,r)}$. Alternatively we say C_j^s is on the boundary of $\text{Can}_i^{(p,r)}$.

Definition 7. A cone with vertex s as endpoint is empty if no edge of E_A or E_{CAN} incident to s is in the cone.

3.2 Cones in Neighbourhoods

When referring to an angle formed by three points, we refer to the smaller of the two angles (that is, the angle that is $< \pi$) unless otherwise stated. When referring to a circle through three points p_1, p_2 , and p_3 , we use the notation O_{p_1, p_2, p_3} .

We consider the edge (p, r) of E_A , where without loss of generality, r is in C_0^p . In this section we show the location of cones in the region of $\text{Can}_0^{(p,r)}$, so we may charge edges of E_{CAN} to them. To facilitate this we introduce another variation on the concept of the neighbourhood of a vertex:

Definition 8. Consider the cone neighbourhood N_i^p with the vertex set $\{p, q_0, q_1, \dots, q_{m-1}\}$, where $\{q_0, q_1, \dots, q_{m-1}\}$ are listed in clockwise order around p . A restricted neighbourhood $N_p^{(q_j, q_k)}$ is the subgraph of N_i^p induced on the vertex set $\{p, q_j, q_{j+1}, \dots, q_k\}$, $0 \leq j \leq k \leq m-1$.

Now we illustrate some of the geometric properties of restricted neighbourhoods in $DT(P)$.

Lemma 1. Consider the arbitrary restricted neighbourhood $N_p^{(r,q)}$. Each vertex $x \in N_p^{(r,q)} \setminus \{p, r, q\}$ is in the circle $O_{p,r,q}$ through p , r , and q .

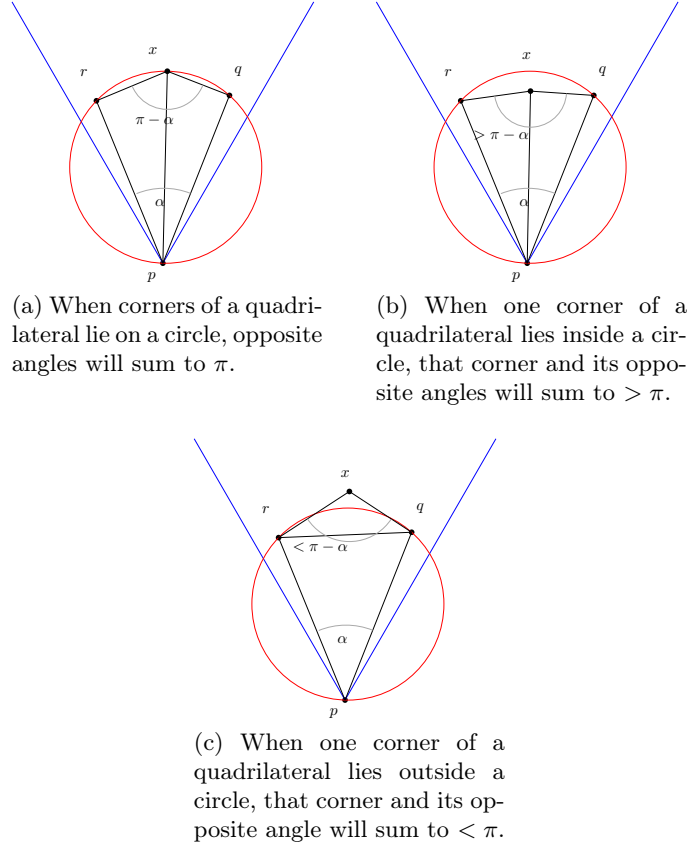


Fig. 6: Properties of convex quadrilaterals in $DT(P)$.

Proof. Since (p, x) is an edge in $DT(P)$, we can draw a disk through p and x that is empty of points of P . In particular, neither r nor q is in this disk. Hence the sum of the angles $\angle(prx)$ and $\angle(pqx)$ which lie on opposite sides of the same chord is smaller than π , and the sum of the other two angles $\angle(rqx)$ and $\angle(rpq)$ in the quadrilateral $(prxq)$ is greater than π . That implies x is inside $O_{p,r,q}$.

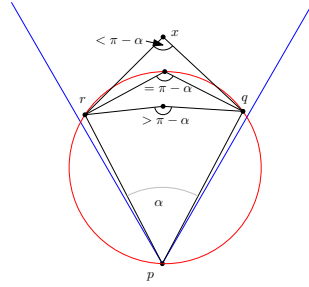
Lemma 2. Consider the restricted neighbourhood $N_p^{(r,q)}$ in cone C_i^p . Let (p, x) be an edge in $N_p^{(r,q)}$ where $x \neq r$ and $x \neq q$. Then angle $\angle(qxr) \geq \pi - \angle(qpr)$. Since the cone angle is $\pi/3$, we have that $\angle(qxr) > 2\pi/3$.

Proof. We know by Lemma 1 that x lies inside the circle through p , r and q , which we label $O_{p,r,q}$. The angle $\angle(qxr)$ is minimized when x is on $O_{p,r,q}$. When x is on $O_{p,r,q}$, $\angle rxq = \pi - \angle(qpr)$, since the two angles lie on the same chord (r, q) . Therefore $\angle(rxq) \geq \pi - \angle(qpr)$. Since both q and r are in the same cone C_i^p , and the cone angle is $\pi/3$, the $\angle(qxr) > 2\pi/3$.

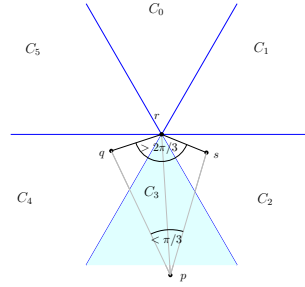
Which leads to the corollary:

Corollary 1. *Let s be an inner vertex of $Can_i^{(p,r)}$ that is not the anchor. Then there is at least one empty cone of s in $Can_i^{(p,r)}$.*

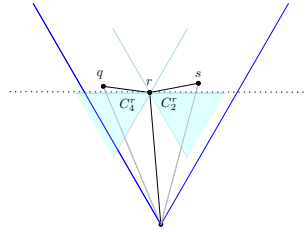
Proof. Since s is not the anchor, any internal cone of $Can_i^{(p,r)}$ on vertex s is empty, and by Lemma 2, there is at least one internal cone of $Can_i^{(p,r)}$ on vertex s . Therefore there is at least one empty internal cone on s in the region of $Can_i^{(p,r)}$. See Fig. 7b. \square



(a) Properties of convex quadrilaterals in $DT(P)$.



(b) Consecutive canonical edges have an angle facing p of at least $2\pi/3$, and thus if $(p, r) \notin E_A$, there is at least one empty cone between them in $D8(P)$.



(c) Lemma 3.

Fig. 7: Locating empty cones.

Lemma 3. *Consider the edge (p, r) in E_A , and without loss of generality let r be in C_0^p . If r is an inner anchor of $\text{Can}_0^{(p,r)}$, then cones C_2^r and C_4^r are empty and in the region of $\text{Can}_0^{(p,r)}$. If r is an end vertex and not the only vertex in $\text{Can}_0^{(p,r)}$, then at least one of C_2^r and C_4^r are empty and in the region of $\text{Can}_0^{(p,r)}$.*

Proof. Since (p, r) is in C_0^p , it must also be in C_3^r , and thus it is in neither C_2^r or C_4^r .

If r is an inner vertex, assume that q, r and s are in consecutive order in $\text{Can}_0^{(p,r)}$. Thus $\text{Can}_i^{(p,r)}$ contains canonical edges (q, r) and (r, s) .

Recall that for every vertex x in $\text{Can}_0^{(p,r)}$, $[px] \geq [pr]$. Thus $[ps] \geq [pr]$ and $[pq] \geq [pr]$, which means that s and q are above the horizontal line through r in C_0^p . Since C_2^r and C_4^r lie below the horizontal line through r , they cannot contain the edges (q, r) and (r, s) . Since $\triangle(pqr)$ and $\triangle(prs)$ are triangles in $DT(P)$, C_2^r and C_4^r are empty and inside $\text{Can}_0^{(p,r)}$. See Fig. 7c.

Otherwise r is an end vertex. By the same argument as above, but applied to only one side of (p, r) , either C_2^r or C_4^r is empty. \square

3.3 Cones in Shared Triangles

We will show the location of uncharged cones in the special case of overlapping regions. A set of regions overlap when at least one triangle of $DT(P)$ is contained in the intersection of all the regions in the set.

Lemma 4. *Consider the triangle $\triangle(pp's)$ in $DT(P)$. Let p' and s be in C_0^p , and let p and s be in $C_3^{p'}$. Then $\angle(p'sp) > 2\pi/3$.*

Proof. Without loss of generality, assume that s is left of directed line segment (p, p') . Consider the parallelogram formed by $C_0^p \cap C_3^{p'}$. Let a be the left intersection and b be the right intersection of C_0^p and $C_3^{p'}$. Thus s is in $\triangle(app')$. Note that $\angle(pp'a) + \angle(app') = \pi/3$. Thus $\angle(pp's) + \angle(spp') < \pi/3$, which implies that $\angle(p'sp) > 2\pi/3$. See Figure 8.

Lemma 5. *Let $\triangle(pp's)$ be a triangle in $DT(P)$. Then $\triangle(pp's)$ can belong to cone neighbourhoods of at most two of p, p' and s .*

Proof. Suppose, for the sake of contradiction, that triangle $\triangle(pp's)$ is in a cone neighbourhood of p , and a cone neighbourhood of p' , and a cone neighbourhood of s . Without loss of generality, let p' and s be in C_0^p . This means that p' and s must be in $C_3^{p'}$. By Lemma 4 angle $\angle(p'sp) > 2\pi/3$. Therefore p and p' cannot be in the same cone neighbourhood of s .

Corollary 2. *A triangle can be shared by at most 2 cone neighbourhoods.*

Proof. Follows from Lemma 5.

Which leads to the following definition:

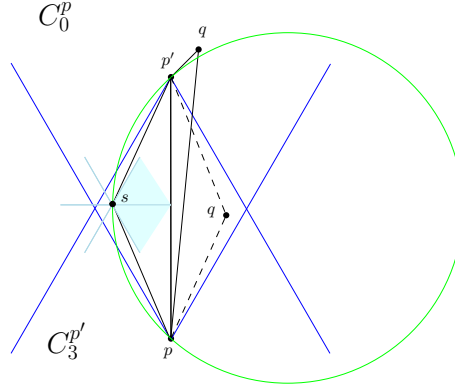


Fig. 8: $q \in C_0^p \cap C_3^{p'}$ violates the empty circle property of Delaunay triangulations.

Definition 9. If $\triangle(pp's)$ occurs in exactly two cone neighbourhoods of p, p' and s , then we refer to it as a shared triangle. If $\triangle(pp's)$ is in cone neighbourhoods of p and p' , then (p, p') is referred to as the base of the shared triangle.

Corollary 3. In a shared triangle $\triangle(pp's)$ with base (p, p') , s has two empty cones internal to $\triangle(pp's)$.

Proof. Without loss of generality, assume that $p' \in C_0^p$ and $p \in C_3^{p'}$. Then $p \in C_3^s$ and $p' \in C_0^s$. Then either C_2^s and C_1^s , or C_5^s and C_4^s are internal to $\triangle(pp's)$ and thus cannot contain any edges with endpoint s .

We show two cone neighbourhoods can share at most one triangle:

Lemma 6. Each Delaunay edge is the base of at most 1 shared triangle.

Proof. By contradiction.

Consider the shared triangle $\triangle(pp's)$ with base (p, p') . Without loss of generality, assume p' is in C_0^p , and p is in $C_3^{p'}$. Since (p, p') is an edge in exactly two triangles, let $\triangle(pp'q)$ be the other triangle with edge (p, p') , and assume that q is in both C_0^p and $C_3^{p'}$.

By Lemma 4 both angles $\angle(p'sp)$ and $\angle(pqp')$ are greater than $2\pi/3$. Thus their sum is greater than $4\pi/3$. But by the empty circle property of Delaunay triangulations, $\angle(p'sp) + \angle(pqp')$ must be less than π , which is a contradiction.

Corollary 4. Two cone neighbourhoods can share at most one triangle in $DT(P)$.

3.4 Empty Cones

Lemma 7. Consider the edge (p, r) in E_A . Without loss of generality let r be in C_0^p . Each inner vertex s of $\text{Can}_0^{(p,r)}$ that is not the anchor has at least one unique empty cone in the region of $\text{Can}_0^{(p,r)}$.

Proof. If s is not part of a shared triangle, we know by Corollary 1 that s has an empty cone internal to $Can_0^{(p,r)}$.

Consider the shared triangle $\triangle(pp's)$, and without loss of generality, let p' be in C_0^p . Assume that there is an edge (p, r) of E_A in C_0^p , and an edge (p', r') of E_A in $C_3^{p'}$. Thus both sets $Can_0^{(p,r)}$ and $Can_3^{(p',r')}$ are well-defined.

By Corollary 3 there are two empty cones of s internal to $\triangle(pp's)$. The empty cone adjacent to (p', s) is the empty cone of s in the region of $Can_0^{(p,r)}$, and the empty cone adjacent to (p, s) is the empty cone of s in the region of $Can_3^{(p',r')}$.

Thus any inner vertex s of an arbitrary canonical subgraph $Can_0^{(p,r)}$ that is not the anchor, has a unique empty cone that is in the region of $Can_0^{(p,r)}$.

Lemma 8. *Consider the edge (p, r) in E_A . Without loss of generality let r be in C_0^p . Let $z \neq r$ be an end vertex in $Can_0^{(p,r)}$. By symmetry, let z be the last vertex. Let y be the neighbour of z in $Can_0^{(p,r)}$. If y is in C_5^z , then C_4^z is a unique empty cone internal to the region of $Can_0^{(p,r)}$.*

Proof. Triangle $\triangle(pyz)$ is a triangle in $DT(P)$. Since (p, z) is in C_3^z and (y, z) is in C_5^z , C_4^z will have no edges in $DT(P)$ with endpoint z . Since both E_A and E_{CAN} are subsets of the edges of $DT(P)$, C_4^z will not contain any edges of E_A and E_{CAN} with endpoint s , and thus is empty.

We prove C_4^z is unique to $Can_0^{(p,r)}$ by contradiction. Since C_4^z is inside a triangle of $DT(P)$, it cannot be a boundary cone, thus it must be inside of shared triangle $\triangle(pyz)$. Corollary 3 states that $\triangle(pyz)$ must have two empty cones internal to $\triangle(pyz)$. However, since (p, z) is in C_3^z , and (y, z) is in C_5^z , only C_4^z is an empty internal cone of $\triangle(pyz)$, which is a contradiction.

Lemma 9. *Consider the edge (p, r) in E_A , and without loss of generality let r be in C_0^p (thus r is the anchor of $Can_0^{(p,r)}$). The empty cones of r internal to $Can_0^{(p,r)}$ are unique to $Can_0^{(p,r)}$.*

Proof. By Lemma 3, C_2^r and C_4^r are (possibly) empty cones inside $Can_0^{(p,r)}$. Since the cases are symmetric, we consider C_2^r . Assume s is the neighbour of r in $Can_0^{(p,r)}$ such that C_2^r is inside $\triangle(rsp)$.

If (r, s) is in C_1^r , then C_2^r is the only empty cone inside $\triangle(rsp)$. By Corollary 3 a shared triangle must have two empty cones, thus C_2^r must be unique to $Can_0^{(p,r)}$.

Otherwise, if (r, s) is in C_0^r , then $\triangle(rsp)$ is a shared triangle. Since both C_2^r and C_1^r are empty, we designate C_2^r as belonging to $Can_0^{(p,r)}$, and C_1^r as belonging to $Can_3^{(s,\cdot)}$. Thus C_2^r is unique to $Can_0^{(p,r)}$.

3.5 Charging Edges in E_A

The charging scheme for the edges of E_A is as follows. Consider an edge (p, r) of E_A , where without loss of generality r is in C_0^p and p is in C_3^r . An edge (p, r) of E_A charges C_0^p once and C_3^r once.

Lemma 10. *Each cone of an arbitrary vertex p of the graph $D8(P)$ is charged at most once by an edge of E_A (thus yielding a maximum degree for the graph $G = (P, E_A)$ of 6).*

3.6 Charging Edges in E_{CAN}

Let (p, r) be an edge of E_A , and without loss of generality let $r \in C_0^p$. Let $Can_0^{(p,r)}$ be the subgraph consisting of the ordered subsequence of canonical edges (s, t) of N_0^p in clockwise order around apex p such that $[ps] \geq [pr]$ and $[pt] \geq [pr]$. We call $Can_0^{(p,r)}$ a canonical subgraph.

For edges in E_{CAN} we consider an arbitrary canonical subgraph $Can_i^{(p,r)}$, and without loss of generality let $i = 0$. We note that there are three types of vertices in $Can_0^{(p,r)}$: anchor, inner and end vertices. Thus any edge added to E_{CAN} from $Can_0^{(p,r)}$ will be charged to an inner, end or anchor vertex (refer to Fig 2). We outline the charging scheme below by referencing the steps of $AddCanonical(p, r)$ where edges were added to E_{CAN} .

Step 1: Without loss of generality, let $r \in C_0^p$.

Step 2: If the anchor r is the first or last vertex in $Can_0^{(p,r)}$, and there is more than one edge in $Can_0^{(p,r)}$, then add the edge of $Can_0^{(p,r)}$ with endpoint r to E_{CAN} . See Fig. 4b.

Step 3: *If there are at least three edges in $Can_0^{(p,r)}$, then for every canonical edge (s, t) in $Can_0^{(p,r)}$ that is not the first or last edge in the ordered subsequence of canonical edges $Can_0^{(p,r)}$, we add (s, t) to E_{CAN} .*

The edge (s, t) is charged once to s and once to t . Since the charging scheme is the same for both s and t , without loss of generality we only describe how to charge s .

Charge vertex s : (Steps 1 and 2)

(a) If s is the anchor (thus $s = r$), then by Lemma 3, C_2^r and C_4^r are empty cones inside $Can_0^{(p,r)}$. If t is left of directed line segment pr , charge (r, t) to C_4^r . If t is right of pr , charge (r, t) to C_2^r . See Fig. 10e.

(b) If $s \neq r$ then by Lemma 7, s has an empty cone C_j^s inside $Can_0^{(p,r)}$. Charge (s, t) once to C_j^s . See Figs 10a, 10c, 10d.

Step 4: *Consider the first and last canonical edge in $Can_0^{(p,r)}$. Since the conditions for the first and last canonical edge are symmetric, we only describe how to process the last canonical edge (y, z) . There are three possibilities.*

(a) *If $(y, z) \in C_5^z$, add (y, z) to E_{CAN} . See Fig. 4c.*

(b) *If $(y, z) \in N_4^z$ and N_4^z does not have an edge with endpoint z in E_A , then we add (y, z) to E_{CAN} . See Fig. 4d*

Charge vertex y :

i If y is the anchor, then C_2^y is empty and inside $Can_0^{(p,r)}$ by Lemma 3. Charge (y, z) to C_2^y . Fig. 10e.

- ii Otherwise y is not the first or last vertex in $Can_0^{(p,r)}$, and by Corollary 1 has an empty cone C_j^y inside $Can_0^{(p,r)}$. Charge (y, z) to C_j^y . Fig.s 10a, 10c, 10d.

Charge vertex z :

- iii Step 4a: (y, z) is in C_5^z . By Lemma 8 C_4^z is empty and inside $Can_0^{(p,r)}$. Charge (y, z) to C_4^z . Fig. 10g.
 - iv Step 4b: (y, z) is in C_4^z , and C_4^z does not contain an edge of E_A with endpoint z . Note C_4^z is a boundary cone of $Can_0^{(p,r)}$. Charge (y, z) to C_4^z . Fig. 10f.
- (c) If $(y, z) \in C_4^z$ and there is an edge $(u, z), u \neq y$ of E_A in C_4^z , then there is one canonical edge of z with endpoint y in C_4^z . Label the edge (w, y) and add it to E_{CAN} . See Fig. 4e.

Charge vertex y :

- i If $y = r$, then C_2^y is empty and inside $Can_0^{(p,r)}$ by Lemma 3. Charge (w, y) to C_2^y . Fig. 10e.
- ii Otherwise y is not the first or last vertex in $Can_0^{(p,r)}$, and by Corollary 1 has an empty cone C_j^y inside $Can_0^{(p,r)}$. Charge (w, y) to C_j^y . Fig. 10h.

Charge vertex w :

- iii If $w = u$ ((z, u) in E_A), then C_2^w is empty and inside $Can_4^{(z,u)}$ by Lemma 11. Charge (w, y) to C_2^w . Fig. 10e.
- iv If $w \neq u$, then w is not the first or last vertex in $Can_4^{(z,u)}$, and by Corollary 1 has an empty cone C_j^w inside $Can_4^{(z,u)}$. Charge (w, y) to C_j^w . Fig. 10i.

Step 4(c)iii makes use of the following lemma:

Lemma 11. Assume that on a call to $AddCanonical(p, r)$, where (p, r) is in C_0^p , we add edge (w, y) to E_{CAN} in Step 4c. Let (y, z) be the last edge in $Can_0^{(p,r)}$, and assume that (w, z) is in E_A . Then C_2^w is empty and inside $Can_4^{(z,u)}$.

Proof. To prove this we shall establish that $[yz] \geq [wz]$. This, together with Lemma 3 implies that C_2^w is empty and inside $Can_4^{(z,u)}$.

We prove by contradiction, thus assume that $[wz] > [yz]$. See Fig. 9. This means that $AddIncident(L)$ examined (y, z) before (w, z) , and thus $C_4^z \cap E_A$ was empty of edges with endpoint z when (y, z) was examined by $AddIncident(L)$. Since (y, z) was not added to E_A , y must have had an edge of E_A in C_1^y with endpoint y that was shorter than (y, z) .

Since $\triangle(pyz)$ is a triangle in $DT(P)$, and $p \in C_3^y$, there cannot be an edge with endpoint y in C_1^y clockwise from (y, z) . In the counter-clockwise direction from (y, z) , we have the $\triangle(ywz) \in DT(P)$. However, since $[wz] > [yz]$, (w, y) cannot be in C_1^y . Thus C_1^y contained no edge of E_A with endpoint y when (y, z) was examined by $AddIncident(L)$. Which means if $[wz] > [yz]$, then (y, z) would have been added to E_A by $AddIncident(L)$. But we know (w, z) is in E_A , therefore

it must be that $[yz] \geq [wz]$, which implies that C_2^w is empty and by Lemma 3 is inside $Can_4^{(z,u)}$. \square

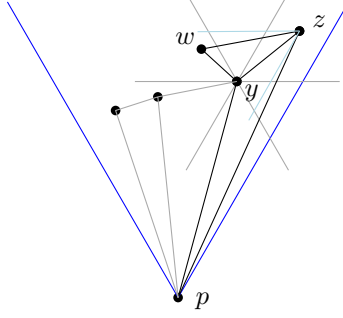


Fig. 9: The case if $[wz] > [yz]$.

3.7 Proving the Degree of D8(P)

The charging argument of the previous section establishes where charges are made. In this section we show a limit to how many edges can be charged to the different cones.

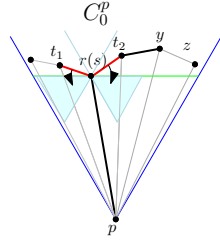
All edges added to E_{CAN} are charged to internal cones, with the exception of the edge that is added in $AddCanonical(p, \cdot)$ Step 4b to C_4^z , which is to a boundary cone. Since C_4^z is on the boundary of $Can_0^{(p,r)}$, it may also be the boundary cone of a different cone neighbourhood.

Lemma 12. *The boundary cone C_4^z of $Can_0^{(p,r)}$ charged in $AddCanonical(p, r)$ Step 4b cannot be the internal cone of a different canonical subgraph.*

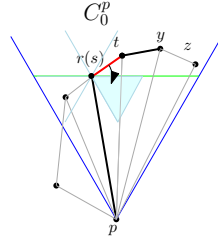
Proof. C_4^z is inside a canonical subgraph, then y must be the apex of said canonical subgraph. That implies that both shared neighbours of z and y must be in C_1^y . But shared neighbour p is in C_3^y , thus C_4^z cannot be inside a canonical subgraph. \square

This implies that C_4^z may only be shared with a different canonical subgraph as a boundary cone. Thus the only other edge of E_{CAN} that can be charged to C_4^z must be added in some call to $AddCanonical(\cdot, \cdot)$ Step 4b. We prove here this is impossible.

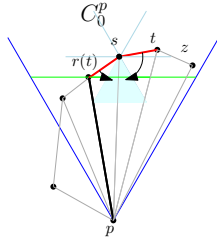
Lemma 13. *Consider the edge (p, r) of E_A in C_i^p , and without loss of generality, let $i = 0$. Let (y, z) be the last edge in $Can_0^{(p,r)}$, and let z be the last vertex in $Can_0^{(p,r)}$. Assume that (y, z) was added to E_{CAN} in a call to $AddCanonical(p, r)$ Step 4b, and thus by Charge iv is charged to the cone C_4^z . Then (y, z) is the only edge in $D8(P)$ charged to C_4^z .*



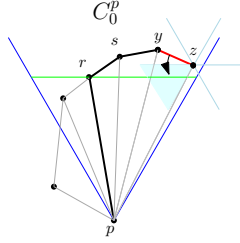
(a) Vertex $r(s)$ is the anchor. Charge (r, t_1) to C_4^s and (r, t_2) to C_2^s .



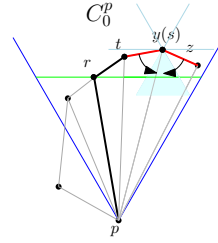
(b) Vertex $r(s)$ is the anchor. Charge (r, t) to C_2^s .



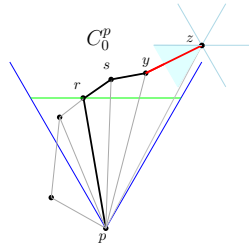
(c) Vertex z is the last vertex of $Can_0^{(p,r)}$, $z \neq r$.



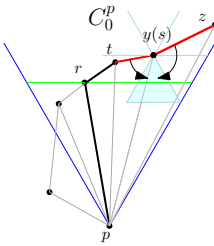
(d) Vertex y is neighbour to the last vertex z .



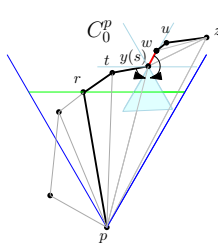
(e) Vertex y charges both edges to its empty cone.



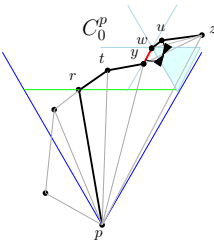
(f) (y, z) is the last canonical edge in $Can_0^{(p,r)}$, and is charged to C_4^z .



(g) (y, z) is the last canonical edge in $Can_0^{(p,r)}$, and is charged to the empty cone of y .



(h) (w, y) is charged to y in place of (y, z) .



(i) (w, y) is charged to the empty cone of w .

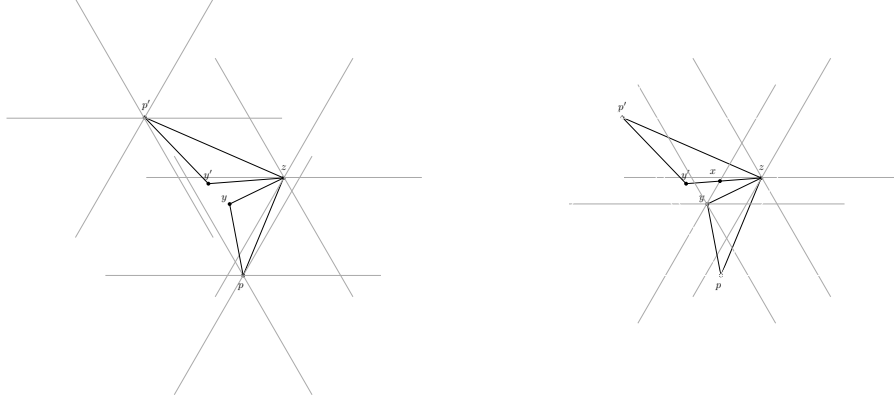
Fig. 10: Charging scheme for edges of E_{CAN} .

Proof. Edge (y, z) is added to E_{CAN} in a call to $AddCanonical(p, r)$ Step 4b, only if there is no edge of E_A in C_4^z . Thus C_4^z is not charged by an edge of E_A .

Cone C_4^z is a boundary cone of $Can_0^{(p,r)}$. By Lemma 12, C_4^z cannot be an internal cone of another canonical subgraph. Thus C_4^z can only be a boundary cone of any canonical subgraph. Since $AddCanonical(\cdot, \cdot)$ Step 4b is the only call that adds an edge to E_{CAN} that is charged to a boundary cone, only another call to $AddCanonical(\cdot, \cdot)$ Step 4b can charge an additional edge to C_4^z .

Assume we have edges (y, z) and (y', z) in $Can_i^{(p,r)}$ and $Can_j^{(p',r')}$ respectively. Without loss of generality, assume that z is the first vertex in $Can_j^{(p',r')}$ and the last vertex in $Can_i^{(p,r)}$, and assume (y, z) and (y', z) occupy the same cone C_k^z . For both (y, z) and (y', z) to be added in $AddCanonical(\cdot, \cdot)$ Step 4b, it must be that $k = i - 2 = j + 2$. Without loss of generality let $i = 0, k = 4$ and $j = 2$, and z is in C_0^p and $C_2^{p'}$.

We know that both $\triangle(pyz)$ and $\triangle(p'y'z)$ are triangles in $DT(P)$. Since $(y, z) \in C_1^y$, and $(y, p) \in C_3^y$, there is no edge in C_1^y clockwise from (y, z) . Symmetrically, there is no edge in $C_1^{y'}$ counter-clockwise from (y', z) . Thus y must have a neighbour closer than z in C_1^y counter-clockwise from (y, z) , and y' must have a neighbour closer than z in $C_1^{y'}$ clockwise from (y', z) . See Figure 11a.



(a) There must be an edge of E_A between (y, z) and (y', z) .

(b) If (y, z) is not in E_A , there must be a neighbour of y or z in $\triangle(yxz)$.

Fig. 11: Lemma 13

We consider the shorter of (y, z) and (y', z) , ties broken arbitrarily. Without loss of generality we will assume $[yz] < [y'z]$. We therefore know that $y' \notin C_1^y$. So there must be a vertex t in C_1^y that is a neighbour of y and closer to y than z counter-clockwise from (y, z) . Since $z \in C_1^y$ and y' is not, the counter-

clockwise cone boundary of C_1^y must intersect (y', z) at a point which we will call x . Therefore t must be in triangle $\triangle(xyz)$. See Figure 11b.

Within $\triangle(xyz)$ we take the closest vertex to z and call it u . (u, z) must be an edge in $DT(P)$, and C_1^u is bounded on one side by (y, z) , and bounded on the other side by (y', z) , and thus z is the closest point to u in C_1^u . Which means that (u, z) would have been added to E_A in $AddIncident()$, which means that C_4^z has an edge $(u, z) \in E_A$. Since there is an edge of E_A in C_4^z , neither (y, z) nor (y', z) would have been added to E_{CAN} in calls to $AddCanonical(\cdot, \cdot)$ Step 4b, and neither would be charged to C_4^z .

This leads to the following corollary:

Corollary 5. *Assume an edge (y, z) is added to E_{CAN} in $AddCanonical(p, r)$ Step 4b, and charged to a boundary cone C_4^z . Then of all the edges in $D8(P)$, only (y, z) is charged to C_4^z .*

The shared triangle is the only scenario where the internal cones of two separate cone neighbourhoods are adjacent on the same vertex.

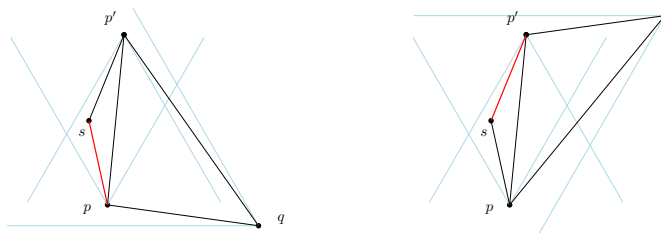
Consider a shared triangle $\triangle(pp's)$ with base (p, p') , and assume that p' is in C_0^p , $(p, r) \in E_A$ is in C_0^p , and $(p', r') \in E_A$ is in $C_3^{p'}$. Vertex s has adjacent cones inside $Can_0^{(p, r)}$ and $Can_3^{(p', r')}$. We prove a limit on the number of canonical edges of p and p' that were added to E_{CAN} and charged to cones of s inside $Can_0^{(p, r)}$ and $Can_3^{(p', r')}$.

Lemma 14. *If (s, p') was added to E_{CAN} and charged to the empty cone of s inside $Can_0^{(p, r)}$, then (s, p) will not be charged to the empty cone of s inside $Can_3^{(p', r')}$.*

Proof. Assume that (p', s) was added to E_{CAN} by a call to $AddCanonical(p, r)$. That implies that (p', s) is not the first or last edge of $Can_i^{(p, r)}$. Thus we know by Lemma 6 that (p, s) must be the last edge in $Can_3^{(p', r')}$, which implies that it is not added by $AddCanonical(p', r')$.

Otherwise assume that (p, p') is a canonical edge of q , and (p, s) was added to E_{CAN} on a call to $AddCanonical(q, \cdot)$ in Step 4c. This implies that (p, p') is in C_5^q . See Figure 12a. There are two possible ways to add (p', s) so that it is charged to the cone of s inside $Can_3^{(p', r')}$. We show that neither occurs:

1. $AddCanonical(q, \cdot)$ adds (p', s) to E_{CAN} in Step 4c. This implies that (p, p') is in C_4^q . See Figure 12b. However, (p, s) was added in Step 4c, which means that (p, p') is in C_5^q , which is a contradiction. Thus both edges cannot be added by calls to $AddCanonical(q, \cdot)$, Step 4c.
2. $AddCanonical(p, r)$ adds (p', s) to E_{CAN} . The shared neighbour q of p and p' is not in C_0^p , and thus (p', s) is the last canonical edge in $Can_0^{(p, r)}$. Thus (p', s) is not added to E_{CAN} by a call to $AddCanonical(p, r)$ (by omission, Step 4).



(a) Only (p, s) can be added to E_{CAN} in Step 4c with apex q .

(b) Only (p', s) can be added to E_{CAN} in Step 4c with apex q .

Fig. 12: The limit on edges added to E_{CAN} in a shared triangle.

Corollary 6. *If the empty cone of s inside $Can_0^{(p,r)}$ is charged twice by edges of $Can_0^{(p,r)}$, then the empty cone of s inside $Can_3^{(p',r')}$ is charged at most once by edges of $Can_3^{(p',r')}$.*

Lemma 15. *All cones charged in the charging scheme are unique to their referenced canonical subgraph.*

Proof. We note that all the edges added here are from a canonical subgraph, thus all the charges are to vertices of a canonical subgraph, and thus must be to an inner vertex, an anchor, or an end vertex. By Lemma 7 each inner vertex has an empty cone unique to its canonical subgraph. By Lemma 8, if the end vertex has an empty cone it is unique to its canonical subgraph, and by Lemma 9, the two possible empty cones on an anchor are unique to its canonical subgraph.

If an edge is added to E_{CAN} in $AddCanonical(p, r)$ Step 4b it is charged to the boundary cone that it occupies. By Lemma 13 it is the only edge charged to that cone, thus we consider it unique to its canonical subgraph.

Lemma 16. *Cones of an end vertex or anchor of a canonical subgraph are charged at most once by edges of E_{CAN} .*

Proof. Lemma 15 proves that all cones charged in the charging scheme are unique (to the referenced canonical subgraph). Since cones of end vertices or anchors are charged at most once in the charging scheme, this implies the lemma.

Lemma 17. *Cones on an inner vertex of a canonical subgraph are charged at most twice by edges of E_{CAN} .*

Proof. Lemma 15 proves that all cones charged in the charging scheme are unique (to the referenced canonical subgraph). Since cones of inner vertices are charged at most twice in the charging scheme, this implies the lemma. \square

Lemma 18. *The edges of E_A and E_{CAN} are never charged to the same cone.*

Proof. The edges of E_A are charged directly to the cone they occupy on each endpoint. We know from the charging scheme that the edges of E_{CAN} are charged to either empty cones, or to a cone that does not contain an edge of E_A . Thus the edges of E_{CAN} and E_A are never charged to the same cone. \square

Lemma 19. *Consider a cone C_i^s of a vertex s in $D8(P)$ that is charged twice by edges of E_{CAN} . Then the two neighbouring cones C_{i-1}^s and C_{i+1}^s are charged at most once by edges of $D8(P)$.*

Proof. Lemmas 10, 16, 17, and 18 state that only a cone on an inner vertex may be double charged.

Each cone C_{i-1}^s and C_{i+1}^s is either an empty internal cone of $Can_i^{(p,r)}$, or a boundary cone containing a canonical edge of $Can_i^{(p,r)}$ with endpoint s . We will consider C_{i+1}^s since the other cases are symmetric.

If C_{i+1}^s is an empty internal cone of $Can_i^{(p,r)}$, then it is only charged for an edge if s is on a shared triangle $\triangle(pp's)$ and s is not on the base. In this case C_{i+1}^s is charged for at most one edge of E_{CAN} by Lemma 6.

Otherwise C_{i+1}^s contains a canonical edge in $Can_i^{(p,r)}$. By our charging scheme and Lemma 6 we know only empty cones are double charged, and by Lemma 18 no cone is charged for both an edge of E_A and an edge of E_{CAN} . Thus C_{i+1}^s is either charged for an edge of E_A , an edge of E_{CAN} , or it is not charged. \square

Theorem 1. *The maximum degree of $D8(P)$ is at most 8.*

Proof. Each edge (p, r) of E_A is charged once to the cone of p containing r and once to the cone of r containing p . By Lemma 10, no cone is charged more than once by edges of E_A .

No edge of E_{CAN} is charged to a cone that is charged by an edge of E_A by Lemma 18.

By Lemma 19, if a cone of a vertex s of $D8(P)$ is charged twice, then its neighbouring cones are charged at most once. This implies that there are at most 3 double charged cones on any vertex s in $D8(P)$.

Assume that we have a vertex s with 3 cones that have been charged twice. A cone of s that is charged twice is an internal cone of some cone neighbourhood N_i^p by our charging argument. Thus s is endpoint to two canonical edges (q, s) and (s, t) in N_i^p . Note that $\angle(qst) > 2\pi/3$ by Lemma 2, and this angle contains the cone of s that is charged twice. Thus to have 3 cones charged twice, the total angle around s would need to be $> 2\pi$, which is impossible. Thus there are at most two double charged cones on s , which gives us a maximum degree of 8. See Fig. 13 for an example of a degree 8 vertex. \square

4 D8(P) is a Spanner

We will prove that $D8(P)$ is a spanner of $DT(P)$ with a spanning ratio of $(1 + \frac{\theta}{\sin \theta}) = (1 + \frac{2\pi}{3\sqrt{3}}) \approx 2.21$, thus making it a $(1 + \frac{2\pi}{3\sqrt{3}}) \cdot C_{DT}$ -spanner of the complete geometric graph, where C_{DT} is the spanning ratio of the Delaunay triangulation. As of this writing, the current best bound of the spanning ratio of the Delaunay triangulation is 1.998 [4], which makes $D8(P)$ approximately a 4.42-spanner of the complete graph.

Suppose that (p, q) is in $DT(P)$ but not in $D8(P)$. We will show the existence of a short path between p and q in $D8(P)$. If the short path from p to q consists of the ideal situation of an edge (p, r) of E_A in the same cone of p as q , plus every canonical edge of p from r to q , then we have what we call the *ideal path*. We give a spanning ratio of the ideal path with respect to the *canonical triangle* T_{pq} , which, informally, is an equilateral triangle with vertex p and height $[pq]$. Notice that in our construction, when adding canonical edges to E_{CAN} on an edge (p, r) of E_A , there are times where the first or last edges of $Can_i^{(p,r)}$ are not added to E_{CAN} . In these cases we prove the existence of alternate paths from p to q that still have the same spanning ratio. Finally we prove that the spanning ratio given in terms of the canonical triangle T_{pq} has an upper bound of $(1 + \theta/\sin \theta)|pq|$, where $\theta = \pi/3$ is the cone angle. A canonical triangle T_{pq} is the equilateral triangle with p at one corner, contained in the cone of p that contains q , and has height $[pq]$.

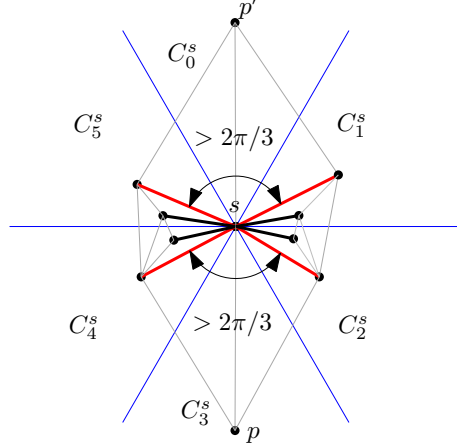


Fig. 13: A degree 8 vertex in $D8(P)$. The red edges belong to E_{CAN} , while the black edges belong to E_A .

4.1 Ideal Paths

We begin by defining the ideal path, and proving the spanning ratio of an ideal path with respect to the graph $DT(P)$.

Definition 10. Consider an edge (p, r) in C_i^p in E_A , and the graph $Can_i^{(p,r)}$. An ideal path is a simple path from p to any vertex in $Can_i^{(p,r)}$ using the edges of $(p, r) \cup Can_i^{(p,r)}$.

Consider an edge (p, r) in C_i^p in E_A , and the graph $Can_i^{(p,r)}$. We will prove that the length of the ideal path from p to q is not greater than $|pa| + \frac{\theta}{\sin \theta}|aq|$, where a is the corner of the canonical triangle to the side of (p, q) that has r , and $\theta = \pi/3$ is the cone angle.

We then use ideal paths to prove there exists a path with bounded spanning ratio between any two vertices p and q in $D8(P)$, where (p, q) is an edge in $DT(P)$. We prove a bound on the length of the path from p to q of $|pa| + \frac{\theta}{\sin \theta} |aq|$.

We note that the distance $|pa| + \frac{\theta}{\sin \theta} |aq|$ is with respect to the canonical triangle T_{pq} rather than the Euclidean distance $|pq|$. To finish the proof we show that $|pa| + \frac{\theta}{\sin \theta} |aq| \leq (1 + \frac{\theta}{\sin \theta}) |pq|$.

To bound the length of ideal paths, we first show that a canonical subgraph forms a path. Then we prove the bound.

We begin with a couple of well-known geometric lemmas. The first is an observation regarding the relative lengths of convex paths, when one resides inside the other.

Lemma 20. *If a convex body C is contained within another convex body C' , then the perimeter of C' is longer than C . [12], page 42.*

The next lemma is a well known result traditionally called “The Inscribed Angle Theorem”.

Lemma 21. *Consider 3 points p, q, s on the boundary of a circle O with center o , such that $\angle(pqs) = \alpha$. Let A be the arc of O from p to s that does not go through q , and let \bar{A} be the arc of O from p to s through q . Then the angle $\angle(pos)$ facing A is equal to 2α . Further, the angles $\angle(pqs)$ facing A is the same for any point p that is on \bar{A} .*

That allows us to establish this result:

Lemma 22. *Let O be a circle through points p and q and r in clockwise order, and let α denote the angle $\angle(qpr)$. Then the length of the arc from q to r on the boundary of $D_{p,q,r}$ is*

$$\frac{\alpha}{\sin \alpha} |qr|$$

Proof. From the center point of O , the angle between q and r is 2α by Lemma 21. Thus the arc length between q and r is $2\alpha R$, where R is the radius of O . Also, $|qr| = 2 \sin \alpha R$, which means $R = \frac{|qr|}{2 \sin \alpha}$. Thus the arc length between q and r is equal to:

$$\begin{aligned} 2\alpha R &= \frac{2\alpha}{2 \sin \alpha} |qr| \\ &= \frac{\alpha}{\sin \alpha} |qr| \end{aligned}$$

which completes the proof. See Figure 14b.

We require that a canonical subgraph is a path, which is proven here.

Lemma 23. *Let (p, r) be an edge in E_A in the cone C_i^p . Then $Can_i^{(p,r)}$ forms a path.*

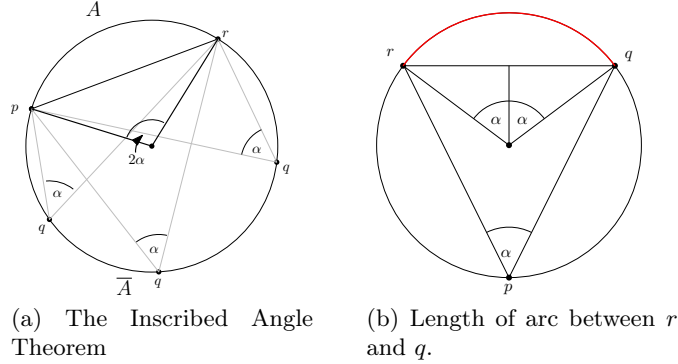


Fig. 14: Relating arc length to angle.

Proof. We prove by contradiction. Note that $Can_i^{(p,r)}$ is a collection of paths. Assume that there are at least two paths in this collection. Without loss of generality, let $i = 0$. Let (a, b) and (y, z) be the first and last edge respectively in $Can_0^{(p,r)}$. Thus of all the vertices in $N_0^p \setminus \{p\}$ between a and z there exists at least one consecutive subset T where for each $t_j \in T, 0 \leq j < |T|, [pt_j] < [pr]$. We consider the vertex $t_k \in T, [pt_k] \leq [pt_j]$, for all $t_j \in T, 0 \leq j < |T|$. Since $[pt_k] < [pr]$, $AddIncident(L)$ examined (p, t_k) before (p, r) . Thus when (p, t_k) was examined, C_i^p contained no edges of E_A with endpoint p . Since (p, t_k) was not added to E_A , there must have been an edge of E_A with endpoint t_k in $C_3^{t_k}$. However, we know $[pt_{k-1}] \leq [pt_k]$ and $[pt_{k+1}] \leq [pt_k]$ (whether or not t_{k-1} and t_{k+1} are in T). Thus neither (t_k, t_{k-1}) nor (t_k, t_{k+1}) can be in $C_3^{t_k}$. Since $\triangle(pt_k t_{k-1})$ and $\triangle(pt_k t_{k+1})$ are triangles in $DT(P)$, the only edge with endpoint t_k in $C_3^{t_k}$ is (p, t_k) . This means that (p, t_k) would have been added to E_A instead of (p, r) , which is a contradiction. \square

Lemma 24. Consider the restricted neighbourhood $N_p^{(r,q)}$ in $DT(P)$ in the cone C_i^p . Let $O_{p,r,q}$ be the circle through the points p, q , and r . Then there are no points of P in $O_{p,r,q}$ to the side of (p, r) that does not contain q . Likewise there are no points of P in $O_{p,r,q}$ to the side of (p, q) that does not contain r .

Proof. Since the cases are symmetric, we prove that there are no points of P in the region R of $O_{p,r,q}$ to the side of (p, r) that does not contain q . We prove by contradiction. Thus assume there is a point t in R . Then the circle $O_{p,t,r}$ contains q and the circle $O_{p,r,q}$ contains t , thus there is no circle through p and r that is empty of points of P . Thus (p, r) cannot be a Delaunay edge, which is a contradiction to our definition of restricted neighbourhood. See Fig. 15a. \square

Lemma 25. Consider the restricted neighbourhood $N_p^{(r,q)}$ in cone C_i^p . Let rq be the directed line from r to q , and assume there are no neighbours of p in $N_p^{(r,q)}$ right of rq . If (r, q) is not an edge in $N_p^{(r,q)}$, then there is a vertex $a \in N_p^{(r,q)}$ such that the circle $O_{r,a,q}$ is empty of vertices of P left of rq .

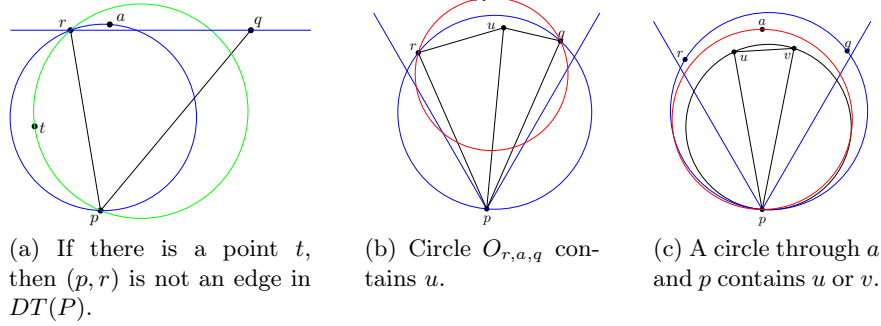


Fig. 15: Locations of a .

Proof. We prove by contradiction, thus assume that we have found a vertex a left of rq such that $O_{r,a,q}$ is empty of vertices of P left of rq , and a is not in $N_p^{(r,q)}$. Note vertex a must exist, otherwise (r, q) is on the convex hull and thus in $N_p^{(r,q)}$. Since the region of $N_p^{(r,q)}$ is empty of vertices of P , a must be outside of $N_p^{(r,q)}$.

We look at two cases:

1. a is outside of $O_{p,r,q}$: Since (r, q) is not in $DT(P)$, there is at least one vertex u in $N_p^{(r,q)} \setminus \{p, r, q\}$. By Lemma 1 and our initial assumption that $N_p^{(r,q)}$ contains no neighbours of p to the right of rq , u must be in $O_{p,r,q}$ to the left of rq . Since a is outside of $O_{p,r,q}$, the arc of $O_{r,a,q}$ to the left of rq contains the arc of $O_{p,r,q}$ to the left of rq . Thus u is in $O_{r,a,q}$ to the left of rq , which is a contradiction to our selection of vertex a . See Fig. 15b.
2. a is inside $O_{p,r,q}$: Since $\angle(rpq) < \pi/3$ (since it is in a cone), and a is inside $O_{p,r,q}$, a must be positioned radially between two consecutive edges with endpoint p in $N_p^{(r,q)}$. Call these edges (p, u) , and (p, v) . Note that $\triangle(puv)$ is a triangle in $DT(P)$, and thus the circle $O_{p,u,v}$ does not contain a by the empty circle property of the Delaunay triangulation. This implies that, since p, u, a , and v form a convex quadrilateral with p and a across the diagonal, any circle through p and a must contain at least one of u or v . Since a is inside $O_{p,r,q}$, $O_{r,a,q}$ contains p . Thus we can draw the circle O_1 through a and p tangent to $O_{r,a,q}$. The portion of O_1 to the left of rq is contained in $O_{r,a,q}$, and thus does not contain any points of P . But any circle through a and p must contain at least one of u and v , and u and v are to the left of rq , which is a contradiction. See Fig. 15c

Thus, if (r, q) is not an edge in $N_p^{(r,q)}$, there is a neighbour a of p in $N_p^{(r,q)}$ such that $O_{r,a,q}$ is empty of vertices of P left of rq . See Fig. 15c.

□

We now turn to a lemma from the paper of Bose and Keil [13] that tells us the length of a path between two points in the Delaunay triangulation of a set

of vertices. We provide a slightly modified and truncated version that suits our needs. The lemma of Bose and Keil does not provide an explicit construction. We apply the lemma to a restricted neighbourhood, and are able to provide a construction of the path along with an upper bound on its length.

Lemma 26. *Consider the restricted neighbourhood $N_p^{(r,q)}$ in $DT(P)$ in the cone C_i^p . Let $\alpha = \angle(rpq) < \pi/3$. If no point of P lies in the triangle $\triangle(prq)$ then there is a path from r to q in $DT(P)$, using canonical edges of p , whose length satisfies:*

$$\delta(r, q) \leq |rq| \frac{\alpha}{\sin \alpha}$$

Proof. Let o be the center of $O_{p,r,q}$, and let $\beta = \angle(roq) = 2\alpha$.

Lemma 24 and the assumption that no vertices of P lie in the triangle $\triangle(prq)$ imply that there are no vertices of P in $O_{p,r,q}$ to the right of directed line segment rq .

We proceed by induction on the number of vertices in $N_p^{(r,q)}$. If there are only 3 vertices in $N_p^{(r,q)}$, then (r, q) is an edge in $DT(P)$, and the path from r to q has length $|rq| < |rq| \frac{\alpha}{\sin \alpha}$ and we are done.

Now assume that the inductive hypothesis holds for all restricted neighbourhoods with fewer vertices than $N_p^{(r,q)}$. Assume $N_p^{(r,q)}$ has more than 3 vertices, otherwise we are done by the same argument as above.

Lemma 24 tells us that there is a vertex a in $N_p^{(r,q)}$ where $O_{r,a,q}$ is empty of vertices of P left of rq .

Let O_1 be the circle through r and a with center o_1 on the line segment (o, r) . Let O_2 be the circle through a and q whose center o_2 lies on the line segment (o, q) . Let $\alpha_1 = \angle(ro_1a)$ and let $\alpha_2 = \angle(ao_2q)$. $N_p^{(r,a)}$ and $N_p^{(a,q)}$ have fewer vertices than $N_p^{(r,q)}$, and O_1 is empty of vertices of P to the right of directed segment ra , and O_2 is empty of vertices of P to the right of directed line segment aq . Thus by the inductive hypothesis:

$$\begin{aligned} \delta(r, q) &= \delta(r, a) + \delta(a, q) \\ &= |ra| \frac{\alpha_1}{\sin \alpha_1} + |aq| \frac{\alpha_2}{\sin \alpha_2} \end{aligned}$$

Let $r' \neq r$ be the intersection of O_1 and rq , and let $q' \neq q$ be the intersection of O_2 and rq . Since $\beta < \pi$, O_1 and O_2 overlap. Let O_3 be the circle through q' and r' with center o_3 on the intersection of the line segment between o_1 and r' and the line segment between o_2 and q' . See Fig. 16.

Triangles $\triangle(roq)$, $\triangle(ro_1r')$, $\triangle(q'o_2q)$, and $\triangle(q'o_3r')$ are all similar isosceles triangles. Thus by Lemmas 21 and 22 the length of the arc of O_1 left of rq is $|rr'| \frac{\alpha}{\sin \alpha}$, the length of the arc of O_2 left of rq is $|q'q| \frac{\alpha}{\sin \alpha}$, and the length of the arc of O_3 left of rq is $|q'r'| \frac{\alpha}{\sin \alpha}$.

Note that O_3 is completely contained in the intersections of O_1 and O_2 . Let A_1 be the arc of O_1 left of rq from a to r' , and let A_2 be the arc of O_2 left of

rq from a to q' . Note that $A_1 \cap A_2$ is a convex shape from q' to r' that contains the arc of O_3 left of rq . Thus $|A_1 \cap A_2| \geq |q'r'| \frac{\alpha}{\sin \alpha}$ by convexity (Lemma 20).

We observe that:

$$\begin{aligned}
\delta(r, q) &= \delta(r, a) + \delta(a, q) \\
&= |ra| \frac{\alpha_1}{\sin \alpha_1} + |aq| \frac{\alpha_2}{\sin \alpha_2} \\
&= |rr'| \frac{\alpha}{\sin \alpha} + |q'q| \frac{\alpha}{\sin \alpha} - |A_1 \cap A_2| \\
&\leq |rr'| \frac{\alpha}{\sin \alpha} + |q'q| \frac{\alpha}{\sin \alpha} - |q'r'| \frac{\alpha}{\sin \alpha} \\
&= |rq| \frac{\alpha}{\sin \alpha}
\end{aligned}$$

as required. □

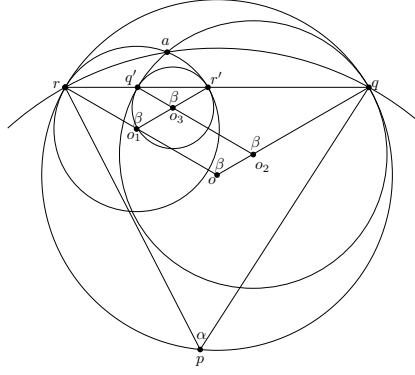


Fig. 16: Lemma 26.

Lemma 27. The path $\delta(r, q) \leq |rr_q| + |r_qq| \frac{\theta}{\sin \theta}$

Proof. By convexity. □

Now we prove the following:

Lemma 28. Consider the restricted neighbourhood $N_p^{(r,q)}$ and without loss of generality let $N_p^{(r,q)}$ be in C_0^p . Let $\alpha = \angle(rpq)$. Let $r_q \neq p$ be the point where the line through p and r intersects the canonical triangle T_{pq} . Let $q_r \neq p$ be the point where the edge (p, q) intersects T_{pr} . If $[pr]$ is the shortest edge of all edges in $N_p^{(r,q)}$ with endpoint p , then the distance from r to q using the canonical edges of p in $N_p^{(r,q)}$ is at most $\max\{|rr_q|, |q_rq|\} + |r_qq| \frac{\theta}{\sin \theta}$.

Proof. Let $\delta(r, q)$ be the length of the path between r and q in $N_p^{(r, q)}$. We will prove by induction on the number of canonical edges of p in $N_p^{(r, q)}$.

If there is only one canonical edge of p in $N_p^{(r, q)}$, then (r, q) is that edge and $\delta(r, q) = |rq| \leq \max\{|rr_q|, |q_rq|\} + |r_qq| \frac{\alpha}{\sin \alpha}$, we are done.

Otherwise assume there is more than one canonical edge of p in $N_p^{(r, q)}$. Consider the edge $(p, a) \in N_p^{(r, q)}$, such that $[pa] \leq [pt]$, for all $(p, t) \in N_p^{(r, q)} \setminus \{r, q\}$. We consider two cases:

1. If $[pa] > [pq]$, then $[pr]$ and $[pq]$ are the shortest edges in $N_p^{(r, q)}$, which implies that there are no points in $\triangle(prq)$. Thus from Lemma 27, the length of the path from r to q is at most $|rq| \frac{\theta}{\sin \theta}$. We have

$$\begin{aligned} \delta(r, q) &\leq |rq| \frac{\alpha}{\sin \alpha} \\ &\leq |rr_q| + |r_qq| \frac{\alpha}{\sin \alpha} \end{aligned}$$

by convexity (Lemma 20). $\frac{\alpha}{\sin \alpha}$ is increasing in α , thus $\frac{\alpha}{\sin \alpha} \leq \frac{\theta}{\sin \theta}$. Thus

$$\begin{aligned} \delta(r, q) &\leq |rr_q| + |r_qq| \frac{\alpha}{\sin \alpha} \\ &\leq \max\{|rr_q|, |q_rq|\} + |r_qq| \frac{\theta}{\sin \theta} \end{aligned}$$

which satisfies the inductive hypothesis. See Fig. 17a

2. $[pa] < [pq]$. Since $[pr] \leq [pa]$ we can apply the inductive hypothesis on $N_p^{(r, a)}$. Let r_a be the point where the line through p and r intersects the horizontal line through a , and let a_r be the point where the line through p and a intersects the horizontal line through r . See Fig. 17b. Then by the inductive hypothesis:

$$\delta(r, a) = \max\{|rr_a|, |a_r a|\} + |r_a a| \frac{\theta}{\sin \theta}$$

Since $[pa] \leq [pq]$ we can apply the inductive hypothesis on $N_p^{(a, q)}$. Let $a_q \neq p$ be the point where the line through a and p exits T_{pq} , and let $q_a \neq p$ be the point where (p, q) intersects T_{pa} , and let $\alpha_2 = \angle(apq)$. See Fig. 17c. Then by the inductive hypothesis:

$$\delta(a, q) = \max\{|aa_q|, |q_a q|\} + |a_q q| \frac{\theta}{\sin \theta}$$

Note that $|pa_q| \leq \max\{|pr_q|, |pq|\}$. Thus:

$$\begin{aligned}
\delta(r, q) &= \max\{|rr_a|, |a_ra|\} + \max\{|aa_q|, |q_aq|\} + |r_aa|\frac{\theta}{\sin\theta} + |a_qq|\frac{\theta}{\sin\theta} \\
&\leq \max\{|rr_q|, |q_rq|\} + |r_aa|\frac{\theta}{\sin\theta} + |a_qq|\frac{\theta}{\sin\theta} \\
&\leq \max\{|rr_q|, |q_rq|\} + |r_qa_q|\frac{\theta}{\sin\theta} + |a_qq|\frac{\theta}{\sin\theta} \\
&\leq \max\{|rr_q|, |q_rq|\} + |r_qq|\frac{\theta}{\sin\theta}
\end{aligned}$$

as required.

See Fig. 17. □

Using Lemma 28 we can prove the main lemma of this section:

Lemma 29. *Consider the edge (p, r) in E_A , located in Can_i^p , and the associated canonical subgraph $\text{Can}_i^{(p,r)}$. Without loss of generality, assume that $i = 0$. The length of the ideal path from p to any vertex q in $\text{Can}_0^{(p,r)}$ satisfies $\delta(p, q) \leq |pa| + \frac{\theta}{\sin\theta}|aq|$, where a is the corner of T_{pq} such that $r \in \triangle(pqa)$, and $\theta = \pi/3$ is the angle of the cones.*

Proof. (Refer to Fig. 17f.) By Lemma 28 the path from r to q is no greater than $\max\{|rr_q|, |q_rq|\} + |r_qq|\frac{\theta}{\sin\theta}$.

Since $|pr| + \max\{|rr_q|, |q_rq|\} \leq |pa|$ and $|aq| \geq |r_qq|$ we have

$$\begin{aligned}
\delta(p, q) &\leq |pr| + \max\{|rr_q|, |q_rq|\} + |r_qq|\frac{\theta}{\sin\theta} \\
&\leq |pa| + |aq|\frac{\theta}{\sin\theta}.
\end{aligned}$$

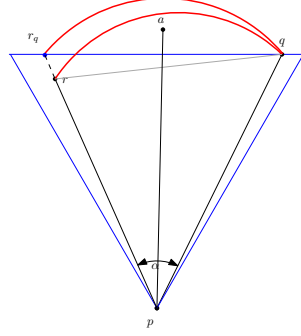
□

4.2 Paths in $D8(P)$

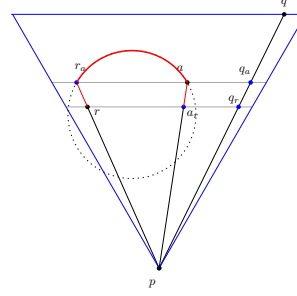
A path in $D8(P)$ that approximates an edge (p, q) of $DT(P)$ can take several forms. It may consist of the edge (p, q) , or it may be an ideal path from p to q , it may be the concatenation of two ideal paths from p to q , or some combination of the above. We prove that $\delta(p, q)$, the length of the path in $D8(P)$ that approximates edge $(p, q) \in DT(P)$, is not longer than $\max\{|pa| + \frac{\theta}{\sin\theta}|aq|, |pb| + \frac{\theta}{\sin\theta}|bq|\}$. Points a and b are the top left and right corners of canonical triangle T_{pq} respectively.

At this point our spanning ratio is with respect to T_{pq} . We then prove that $D8(P)$ is a spanner with respect to the Euclidean distance $|pq|$.

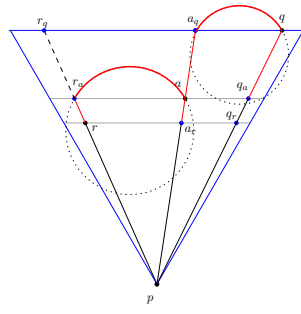
We consider an edge $(p, q) \in DT(P)$. If $(p, q) \in D8(P)$ then the length of the path from p to q in $D8(P)$ is $|pq| \leq (1 + \frac{\theta}{\sin\theta})|pq|$, as required.



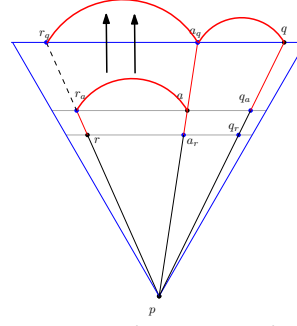
(a) If $|pa| \geq |pq|$, apply Lemma 26.



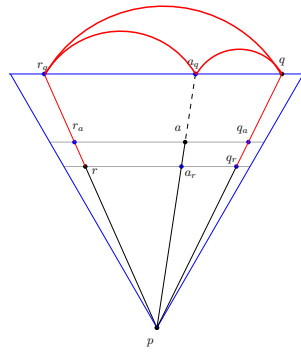
(b) If the vertex a is in $\triangle(prq)$, we proceed by induction from r to a .



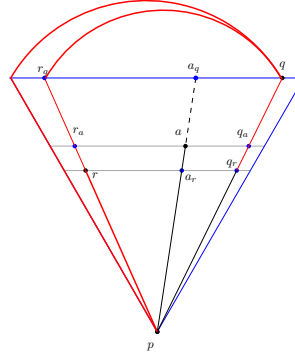
(c) Proceed by induction from a to q .



(d) $|r_a a| \frac{\theta}{\sin \theta} < |r_q a_q| \frac{\theta}{\sin \theta}$.



(e) $|r_q q| \frac{\theta}{\sin \theta} = |r_q a_q| \frac{\theta}{\sin \theta} + |a_q q| \frac{\theta}{\sin \theta}$.



(f) Lemma 29.

Fig. 17: Inductive path.

Thus we assume $(p, q) \notin D8(P)$. Without loss of generality we assume q is in C_0^p . Since $(p, q) \notin D8(P)$, there is an edge (p, r) of E_A in Can_0^p or (q, u) in Can_3^q (or both), where $[pr] \leq [pq]$ and $[qu] \leq [pq]$. Otherwise (p, q) would have been added to E_A in $AddIncident(L)$. Without loss of generality we shall assume there is the edge $(p, r) \in E_A$, $[pr] \leq [pq]$, and that (p, q) is clockwise from (p, r) around p .

Let s be the vertex such that s is a neighbour of q in $N_p^{(p,r)}$ and $s \neq p$ (but possibly $s = r$). Let a be the upper left corner of T_{pq} , and b be the upper right corner. Let $\alpha = \angle(rpq)$ and $\theta = \pi/3$ be the angle of the cones.

Lemma 30. *Recall that $(p, r) \in E_A$, where $r \in C_i^p$. Then there is an ideal path from p to any vertex q in $Can_i^{(p,r)}$, where q is not an end vertex of Can_i^p .*

Proof. In the algorithm $AddCanonical(p, r)$, we add every canonical edge of p in $Can_i^{(p,r)}$ that is not the first or last edge. By Lemma 23, the edges of $Can_i^{(p,r)}$ form a path. Thus there is the ideal path from p to any vertex q in $Can_i^{(p,r)}$ that is not the first or last vertex. \square

The next lemmas prove that, for a vertex z that is the first or the last vertex of Can_i^p , the edge in Can_i^p with endpoint z cannot be in C_i^z .

Lemma 31. *Let r and q be two consecutive neighbours of p , in an arbitrary cone C_i^p . Without loss of generality, let (p, q) be clockwise from (p, r) in the cone C_i^p . If q is in C_i^r , then all edges with endpoint p in C_i^p that appear after (p, q) in clockwise order are longer than $[pq]$.*

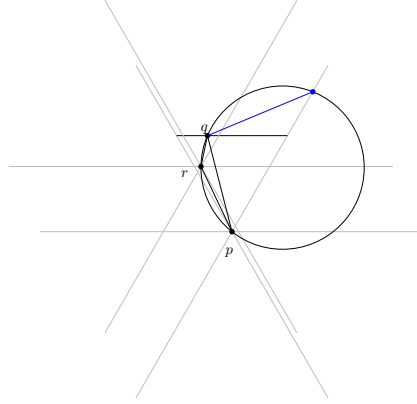
Proof. By Lemma 2, any edge (p, t) clockwise from (p, q) in C_i^p is such that the angle $\angle(rqt) > 2\pi/3$. Since (r, q) is in C_i^r , it is at an angle of at least $\pi/3$ from the positive x -axis. Since $\angle(rqt) > 2\pi/3$, the edge (r, t) must be at an angle > 0 with respect to the positive x -axis. Thus $[pt] > [pq]$, for all (p, t) clockwise from (p, q) in C_i^p . See Figure 18a.

Lemma 32. *Let z be the first or last vertex of $Can_i^{(p,r)}$, and assume that (p, z) is not in E_A . Let (y, z) be the last edge in $Can_i^{(p,r)}$. Then (y, z) is not in C_i^z .*

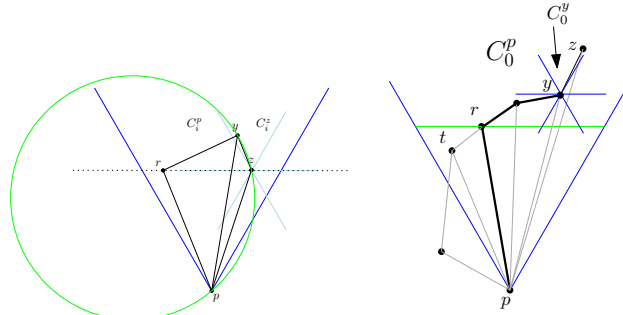
Proof. We assume that $(y, z) \in C_i^z$, and prove by contradiction. By Lemma 31, if (y, z) is in C_i^z , then (p, y) is the shortest of all edges in C_i^p with endpoint p counter-clockwise from (p, y) .

Let (p, r) be an edge in E_A , where $r \in Can_i^{(p,r)}$. Then (p, r) is at least as short as all edges in $DT(P)$ from p to a vertex in $Can_i^{(p,r)}$. But that is a contradiction to (p, y) (and by extension (p, z)) being the shortest. See Fig. 18b \square

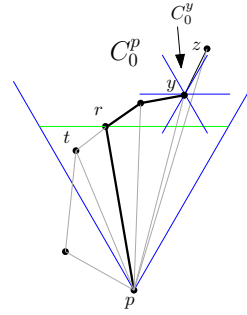
Let (p, r) be an edge in E_A is the graph $D8(P)$. Without loss of generality, assume that r is in C_0^p . By Lemma 30, there is the ideal path from p to any vertex in $Can_0^{(p,r)}$ that is not the first or last vertex. We now turn our attention to the first or last vertex in $Can_0^{(p,r)}$. Because the cases are symmetric, we focus



(a) Any edge with endpoint p in N_i^p clockwise from (p, q) must be longer than (p, q) .



(b) (y, z) cannot be in C_i^z .



(c) If the final edge of Can_0^p is in C_0^y , we do *not* add it to E_{CAN} .

Fig. 18: The edge in Can_i^p with endpoint z cannot be in C_i^z .

on the last vertex, which we designate z . If $z = r$, the path from p to z is trivial, thus we assume $z \neq r$. Let y be the neighbour of z in $Can_0^{(p,r)}$. By Lemma 32, (y, z) cannot be in C_0^z . Thus (y, z) can be in C_5^z , C_4^z , or C_3^z .

Case 1: Edge (y, z) is in C_5^z . Then (y, z) was added to E_{CAN} in $AddCanonical(p, r)$, Step 4a, and there is an ideal path from p to z .

Case 2: Edge (y, z) is in C_4^z . There are three possibilities.

- (a) If (y, z) is an edge of E_A , then there is an ideal path from p to z .
- (b) If there is no edge in E_A with endpoint z in C_4^z , then (y, z) was added to E_{CAN} in $AddCanonical(p, r)$, Step 4b, and there is an ideal path from p to z .
- (c) If there is an edge of E_A in C_4^z with endpoint z that is not (y, z) , then we have added the canonical edge of z in C_4^z with endpoint y to E_{CAN} in $AddCanonical(p, r)$, Step 4c. Therefore by Lemma 30 there is an ideal path from z to y , and also an ideal path from p to y .

Case 3: Edge (y, z) is in C_3^z . Then (y, z) was not added to E_{CAN} .

In Case 1, Case 2a, and Case 2b there is an ideal path from p to q . Thus Lemma 29 tells us there is a path from p to q not longer than $|pa| + \frac{\theta}{\sin \theta} |aq|$.

In Case 2c, we have two ideal paths that meet at y . As in the case of a single ideal path, the sum of the lengths of these two paths is not more than $|pa| + \frac{\theta}{\sin \theta} |aq|$. The following lemma proves this claim:

Lemma 33. *Consider the edge (p, r) in E_A in the graph $D8(P)$, r in C_0^p . Let (y, z) be the last edge in $Can_0^{(p,r)}$, and let (y, z) be in C_4^z . Let (z, u) be an edge in E_A in C_4^z . Assume there is an ideal path from p to y in C_0^p , and an ideal path from z to y in C_4^z . Let a be the top left corner of T_{pz} . We prove an upper bound on the length $\delta(p, z)$ of $|pa| + \frac{\theta}{\sin \theta} |az|$.*

Proof. Let a_1 be the top left corner of T_{py} , and let b_2 be the top right corner of T_{zy} (as seen from apex z . Note that T_{zy} lies in C_z^4). Since (y, z) is the last edge in $Can_0^{(p,r)}$, we note that the ideal path from p to y is to the side of (p, y) that contains r and does not contain z . Similarly, the ideal path from z to y is to the side of (y, z) that contains u and does not contain p . See Figure 19. By Lemma 30, the length of the path from p to z in $D8(P)$ is:

$$\begin{aligned}
\delta_{D8(P)}(p, z) &\leq \delta_{D8(P)}(p, y) + \delta_{D8(P)}(z, y) \\
&\leq |pa_1| + \frac{\theta}{\sin \theta} |a_1 s| + |zb_2| + \frac{\theta}{\sin \theta} |b_2 y| \\
&\leq |pa_1| + \frac{\theta}{\sin \theta} |a_1 s| + |b_2 y| + \frac{\theta}{\sin \theta} |zb_2| \tag{1} \\
&\leq (|pa_1| + |b_2 y|) + \frac{\theta}{\sin \theta} (|a_1 y| + |zb_2|) \tag{2} \\
&= |pa| + \frac{\theta}{\sin \theta} |az| \tag{3}
\end{aligned}$$

Inequality 1 holds because $\frac{\theta}{\sin \theta} > 1$, and $|b_2y| \leq |zb_2|$, since $|zb_2|$ is the longest possible line segment in T_{zy} .

In Case 3 there is no edge from y to z . We prove the length of the path from p to z in Case 3 by induction, as part of the main lemma of this section:

Lemma 34. *Consider the edge (p, r) in E_A in the graph $D8(P)$. Without loss of generality, let r be in C_0^p . Let a and b be the top left corner and top right corner respectively of T_{pq} . For any edge $(p, q) \in DT(P)$, there exists a path from p to q in $D8(P)$ that is not longer than $\max\{|pa| + \frac{\theta}{\sin \theta}|aq|, |pb| + \frac{\theta}{\sin \theta}|bq|\}$.*

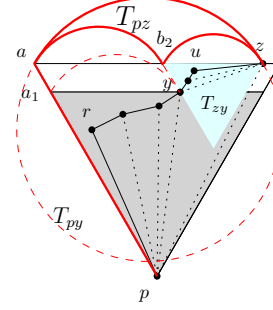


Fig. 19: Concatenating ideal paths.

Proof. Let $\delta(p, q)$ be the shortest path from p to q in $D8(P)$. We do a proof by induction on the size of the canonical triangle T_{pq} .

The base case is when T_{pq} is the smallest canonical triangle. One instance of this occurs when there is an ideal path from p to q , as in Case 1, Case 2a, and Case 2b. Thus by Lemma 29:

$$\delta(p, q) \leq |pa| + \frac{\theta}{\sin \theta}|aq|.$$

The other instance is Case 2c, where two ideal paths meet at a vertex. By Lemma 33 we have:

$$\delta(p, q) \leq |pa| + \frac{\theta}{\sin \theta}|aq|.$$

Since $|aq| \leq \max\{|aq|, |bq|\}$, the proof holds in all base cases.

In Case 3, q is the first or last vertex in $Can_0^{(p,r)}$. Since the cases are symmetric, consider when q is the last vertex, and assume it has a neighbour s in $Can_0^{(p,r)}$, such that the canonical edge (s, q) in N_0^p is in C_0^s . Thus (s, q) was not added to E_{CAN} on a call to $AddCanonical(p, r)$.

We break down T_{pq} into canonical triangles T_{ps} and T_{sq} . Call the upper left corner of T_{pq} a , and the upper right corner b . Also the upper left corner of T_{ps} is a_1 , the upper right corner of T_{sq} is a_2 , the upper right corner of T_{ps} is b_1 , and the upper right corner of T_{sq} is b_2 . Since (s, q) is in C_0^s , both T_{ps} and T_{sq} must be smaller than T_{pq} .

We note the following facts:

- Fact 1: $|pa| = |pa_1| + |sa_2|$ and likewise $|pb| = |pb_1| + |sb_2|$
- Fact 2: $|ab| = |a_1b_1| + |a_2b_2|$
- Fact 3: $|aa_2| = |a_1s|$ and $|bb_2| = |sb_1|$
- Fact 4: q is on the line (a_2, b_2)

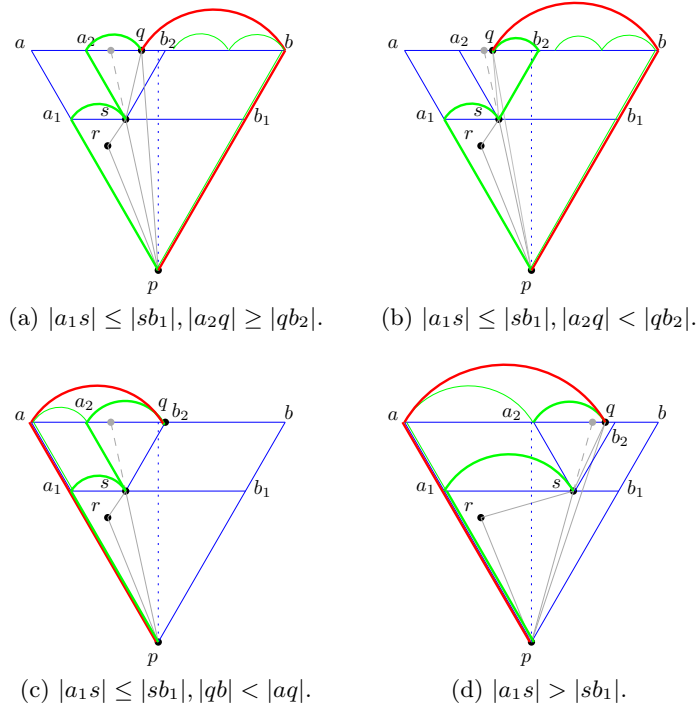


Fig. 20: Dark green are the actual paths, light green demonstrates the path is not longer than the red path.

Without loss of generality, assume the path from p to s is to the side of the line through p and s with a_1 (note that we are not assuming that $|a_1s| > |b_1s|$).

We extend the line (p, s) until it intersects (a_2, b_2) at a point we label s' . Since q is the last vertex in $Can_0^{(p,r)}$, q must be to the side of s' closer to b_2 .

Since $|pa| = |pb|$ and $|sa_2| = |sb_2|$, it is sufficient to prove:

$$|pa_1| + \frac{\theta}{\sin \theta} |a_1s| + |sa_2| + \frac{\theta}{\sin \theta} \max\{|a_2q|, |qb_2|\} \leq |pa| + \frac{\theta}{\sin \theta} \max\{|aq|, |bq|\}$$

By Fact 1 this is equivalent to:

$$\begin{aligned} \frac{\theta}{\sin \theta} |a_1s| + \frac{\theta}{\sin \theta} \max\{|a_2q|, |qb_2|\} &\leq \frac{\theta}{\sin \theta} \max\{|aq|, |bq|\} \\ |a_1s| + \max\{|a_2q|, |qb_2|\} &\leq \max\{|aq|, |qb|\} \end{aligned}$$

We consider two scenarios:

1. $|a_1s| \leq |sb_1|$: There are two sub-cases:
 - (a) $|qb| \geq |aq|$: If $|a_2q| \geq |qb_2|$, then:

$$\begin{aligned} |a_1s| + |a_2q| &\leq |aq| \\ &\leq |qb| \end{aligned}$$

as required. Otherwise, $|qb_2| > |a_2q|$, thus:

$$\begin{aligned} |a_1s| + |qb_2| &\leq |sb_1| + |qb_2| \\ &= |qb| \end{aligned}$$

as required.

- (b) $|qb| < |aq|$: Together with $|a_1s| \leq |sb_1|$ implies that $|a_2q| > |qb_2|$. See Figure 20d. Then $|a_1s| + |a_2q| = |aq|$, as required.
2. $|a_1s| > |sb_1|$: Since q is radially to the right of (p, s) , $|aq| > |qb|$. It is also true that $|a_2q| > |qb_2|$. Thus, using Fact 3:

$$\begin{aligned} |a_1s| + |a_2q| &= |aa_2| + |a_2q| \\ &= |aq| \end{aligned}$$

as required. See Figure 20c.

For an edge (p, q) in $DT(P)$, we have a bound on the length of the path in $D8(P)$. However, this bound is terms of the size of the canonical triangle T_{pq} , which is not the same as the Euclidean distance $|pq|$. In the following section we prove that $\max\{|pa| + \frac{\theta}{\sin \theta} |aq|, |pb| + \frac{\theta}{\sin \theta} |bq|\} \leq (1 + \frac{\theta}{\sin \theta}) |pq|$.

4.3 The Spanning Ratio of D8(P)

Lemma 35.

$$\max\{|pa| + \frac{\theta}{\sin \theta}|aq|, |pb| + \frac{\theta}{\sin \theta}|bq|\} \leq \left(1 + \frac{\theta}{\sin \theta}\right) |pq|$$

Proof. Without loss of generality, we will assume that

$$\max\{|pa| + \frac{\theta}{\sin \theta}|aq|, |pb| + \frac{\theta}{\sin \theta}|bq|\} = |pa| + \frac{\theta}{\sin \theta}|aq|$$

Let

$$\lambda = \left(\frac{\theta}{\sin \theta} - 1\right) (|pq| - |aq|)$$

We will show that:

$$|pa| + \frac{\theta}{\sin \theta}|aq| \leq |pa| + \frac{\theta}{\sin \theta}|aq| + \lambda \leq \left(1 + \frac{\theta}{\sin \theta}\right) |pq|$$

Since $|pq| \geq |pa|$ (by the sine law), and $\frac{\theta}{\sin \theta} > 1$, we get $\lambda \geq 0$. Thus

$$\begin{aligned} & |pa| + \frac{\theta}{\sin \theta}|aq| \\ & \leq |pa| + \frac{\theta}{\sin \theta}|aq| + \lambda \end{aligned}$$

It remains to be shown that:

$$\begin{aligned} & |pa| + \frac{\theta}{\sin \theta}|aq| + \lambda \leq \left(1 + \frac{\theta}{\sin \theta}\right) |pq| \\ & |pa| + \frac{\theta}{\sin \theta}|aq| + \left(\frac{\theta}{\sin \theta} - 1\right) (|pq| - |aq|) \leq \left(1 + \frac{\theta}{\sin \theta}\right) |pq| \\ & |pa| - |pq| + |aq| + \frac{\theta}{\sin \theta}(|aq| + |pq| - |aq|) \leq \left(1 + \frac{\theta}{\sin \theta}\right) |pq| \\ & |pa| - |pq| + |aq| + \frac{\theta}{\sin \theta}|pq| \leq |pq| + \frac{\theta}{\sin \theta}|pq| \\ & |pa| - |pq| + |aq| \leq |pq| \\ & |pa| + |aq| \leq 2|pq| \end{aligned}$$

Thus we must show that $|pa| + |aq| \leq 2|pq|$ holds true for all values of $\alpha = \angle(apq)$.

Let a' be the point to the side of (p, q) that contains a , such that $\triangle(a'pq)$ is an equilateral triangle. Thus

$$|pa'| + |a'q| = 2|pq|.$$

See Fig. 21c. We will prove that

$$|pa| + |aq| \leq |pa'| + |a'q| = 2|pq|.$$

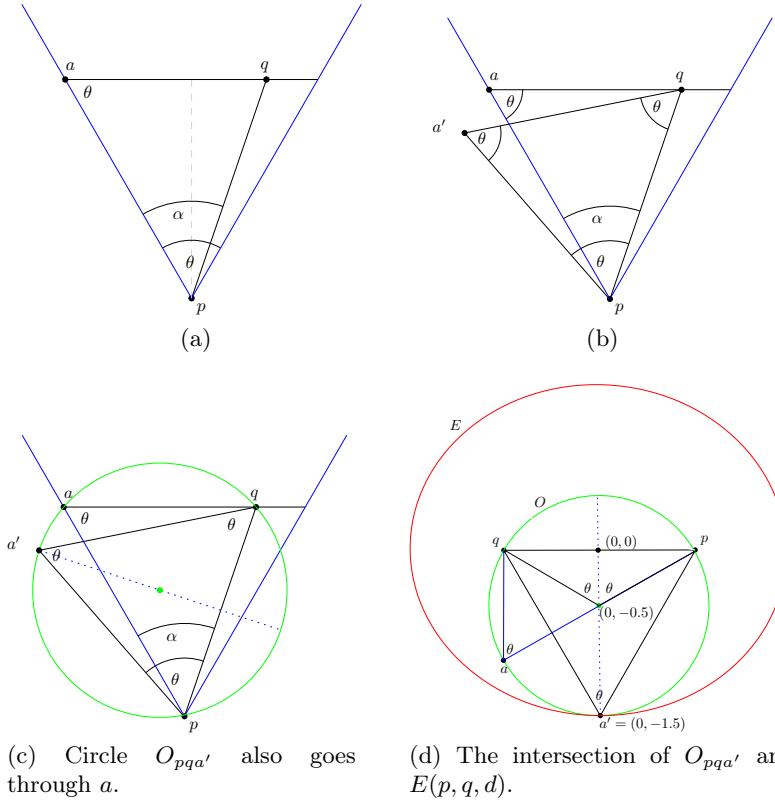


Fig. 21: $|pa| + |aq| \leq |pa'| + |a'q| = 2|pq|$.

Note that $\angle(paq) = \angle(pa'q) = \theta$. That implies that the circle $O_{pa'q}$ through p , a' and q also goes through a . See Fig. 21c.

To better analyze the problem, we rotate, translate, and scale p , q , a and a' such that $q = (-\sin \theta, 0)$, $p = (\sin \theta, 0)$, and $a' = (0, -1.5)$. Let a be any point on $O_{pqa'}$ below the line through p and q . Let $E(p, q, d)$, where $d = 2|pq|$, represent an ellipse with focal points p and q such that for each point b on the boundary

of $E(p, q, d)$, $|pb| + |bq| = d = 2|pq|$. Note the center of $O_{pqa'}$ is $(0, -0.5)$, and $O_{pqa'}$ has a radius of 1. See Fig. 21d. The equation for $O_{pqa'}$ is:

$$x^2 + (y + \frac{1}{2})^2 = 1$$

The equation for $E(p, q, d)$ is:

$$\begin{aligned} \frac{x^2}{a^2} + \frac{y^2}{b^2} &= 1 \\ \frac{x^2}{(2 \sin \theta)^2} + \frac{y^2}{\frac{3}{2}} &= 1 \\ \frac{x^2}{3} + \frac{4y^2}{9} &= 1 \end{aligned}$$

Thus we find the intersection of $O_{pqa'}$ and $E(p, q, d)$ by solving the following system of equations:

$$\begin{aligned} x^2 + (y + \frac{1}{2})^2 &= 1 \\ \frac{x^2}{3} + \frac{4y^2}{9} &= 1 \end{aligned}$$

This gives us a single solution at $(0, -1.5)$.

Note that, when $\angle(aqp) = \pi/2$, $|pa| = 2$ and $|aq| = 2 \cos \theta = 1$. Thus $|pa| + |aq| = 3$. We have $2|pq| = 2 * (2 \sin \theta) \approx 3.46$. Thus when $\angle(aqp) = \pi/2$, $|pa| + |aq| < 2|pq| = |pa'| + |a'q|$, which means that a is inside $E(p, q, d)$, which means all of $O_{pqa'}$ is inside $E(p, q, d)$, with the exception of $(0, -1.5)$. Thus for all points a on $O_{pqa'}$,

$$|pa| + |aq| \leq |pa'| + |a'q| = 2|pq|.$$

Which implies that:

$$\delta(p, q) \leq |pa| + \frac{\theta}{\sin \theta} |aq| \leq (1 + \frac{\theta}{\sin \theta}) |pq|$$

as required. □

Using this inequality and Lemma 34, the main theorem now follows:

Theorem 2. *For any edge $(p, q) \in DT(P)$, there is a path in $D8(P)$ from p to q with length at most $(1 + \frac{\theta}{\sin \theta}) |pq|$, where $\theta = \pi/3$ is the cone width. Thus $D8(P)$ is a $(1 + \frac{\theta}{\sin \theta}) D_T$ -spanner of the complete graph, where D_T is the spanning ratio of the Delaunay triangulation (currently 1.998 [4]).*

References

1. Chew, P.: There is a planar graph almost as good as the complete graph. In: Proceedings of the Second Annual Symposium on Computational Geometry. SCG '86, New York, NY, USA, ACM (1986) 169–177
2. Dobkin, D., Friedman, S., Supowit, K.: Delaunay graphs are almost as good as complete graphs. *Discrete & Computational Geometry* **5** (1990) 399–407
3. Keil, J., Gutwin, C.: Classes of graphs which approximate the complete euclidean graph. *Discrete & Computational Geometry* **7** (1992) 13–28
4. Xia, G.: Improved upper bound on the stretch factor of Delaunay triangulations. In: Proceedings of the Twenty-seventh Annual Symposium on Computational Geometry. SoCG '11, New York, NY, USA, ACM (2011) 264–273
5. Bose, P., Gudmundsson, J., Smid, M.: Constructing plane spanners of bounded degree and low weight. In Mǎřřhring, R., Raman, R., eds.: Algorithms - ESA 2002. Volume 2461 of Lecture Notes in Computer Science. Springer Berlin Heidelberg (2002) 234–246
6. Li, X.Y., Wang, Y.: Efficient construction of low weight bounded degree planar spanner. In Warnow, T., Zhu, B., eds.: Computing and Combinatorics. Volume 2697 of Lecture Notes in Computer Science. Springer Berlin Heidelberg (2003) 374–384
7. Bose, P., Smid, M.H.M., Xu, D.: Delaunay and diamond triangulations contain spanners of bounded degree. *Int. J. Comput. Geometry Appl.* (2009) 119–140
8. Kanj, I.A., Perković, L., Xia, G.: On spanners and lightweight spanners of geometric graphs. *SIAM Journal on Computing* **39** (2010) 2132–2161
9. Bose, P., Carmi, P., Chaitman-Yerushalmi, L.: On bounded degree plane strong geometric spanners. *Journal of Discrete Algorithms* **15** (2012) 16 – 31
10. Bonichon, N., Gavoille, C., Hanusse, N., Perković, L.: Plane spanners of maximum degree six. In Abramsky, S., Gavoille, C., Kirchner, C., Meyer auf der Heide, F., Spirakis, P., eds.: Automata, Languages and Programming. Volume 6198 of Lecture Notes in Computer Science. Springer Berlin Heidelberg (2010) 19–30
11. Bonichon, N., Kanj, I., Perković, L., Xia, G.: There are plane spanners of degree 4 and moderate stretch factor. *Discrete & Computational Geometry* **53** (2015) 514–546
12. Benson, R.: Euclidean geometry and convexity. McGraw-Hill (1966)
13. Bose, P., Keil, J.M.: On the stretch factor of the constrained Delaunay triangulation. In: 3rd International Symposium on Voronoi Diagrams in Science and Engineering, ISVD 2006, Banff, Alberta, Canada, July 2-5, 2006, IEEE Computer Society (2006) 25–31